Elementary Relativistic Wave Mechanics of Spin 0 and Spin 1/2 Particles*

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1. INTRODUCTION

THIS paper presents, from an elementary point of view, a unified picture of the relativistic single particle wave mechanics for both spin 0 boson and the spin $\frac{1}{2}\hbar$ fermion. Most of the results obtained are well known, but in discussion with many of our colleagues we have found that there is considerable confusion about some aspects of relativistic wave mechanics, which arises mainly in connection with the physical interpretation of the formalism. Textbooks and most papers generally content themselves with reproducing the formal mathematical apparatus. Notable exceptions are the papers by Heitler,[1] Kemmer,[2] and Sakata and Taketani on bosons[3] and by Foldy and Wouthuysen[4] on fermions. However, as far as we know, a systematic analysis of the similarities and differences between the theory of the boson and that of the fermion has never been given.

This paper is then frankly pedagogical in nature. The single particle equations discussed are the Klein-Gordon equation for spin 0, and the Dirac equation for spin $\frac{1}{2}\hbar$. Greatest emphasis is placed on questions of interpretation. For example, although it is well known that the Dirac equation gives, within proper limits, a relativistic wave-mechanical description of a single electron, we find in the literature the (incorrect!) stater ment that an analogous formalism does not exist fo- charged spin zero particles. Another set of questions deals with the relativistic description of neutral particles and more generally of charge multiplets (e.g., the π meson). Finally there is the very important question of the relation of one particle relativistic wave mechanics to the more general and powerful quantum field formalism.

In all these instances it is essential to restate the wave equation in such a manner as to make the physical interpretation most transparent. The most significant element in such an interpretation is the definition of a "single particle" in terms of its wave function. For both wave equations it is possible to obtain two independent solutions, one of which has the total charge $+e$, the other $-e$, and these solutions are taken to represent a single particle of the corresponding charge. However, the wave equation must be so formulated as to make this separation into solutions for positive and negative charge an obvious one, as is, for example, not the case in the customary statement of the Klein-Gordon equation using a single complex wave function. In this case it is best to make the charge degree of freedom more visible by using a two-component wave function. The corresponding wave equation is no longer obviously covariant for, as might be expected, interpretative discussion is most easily made when time and space coordinates are distinguished.

We use the spin 0 case rather than the spin $\frac{1}{2}$ case to develop most of our methods principally because this case does not involve any effects of spin and is thus much simpler. The relativistic and spin effects may be

* This work was supported in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

[1] There is a tremendous literature on this subject, and we can hardly hope to do justice to all of it. References are restricted to those papers which we have found useful. W. Heitler, Proc. Roy. Irish Acad. 49, 1 (1943).

[2] N. Kemmer, Proc. Roy. Soc. (London) A173, 91 (1939).

[3] Sakata and Taketani, Proc. Phys. Math. Soc. Japan 22, 757 (1940).

[4] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

clearly distinguished by then comparing the spin zero and spin $\frac{1}{2}$ cases. This order of development impresses on the reader the basic similarity of the two cases and the identical nature of the limitations imposed on the one particle description.

The discussion of the wave mechanics of spin 0 particles is given in Sec. 2. Section 3 deals with the spin $\frac{1}{2}$ case. The discussion in each follows parallel routes to emphasize similarities. In both, the relativistic equations are derived as generalizations of corresponding nonrelativistic equation. This generalization is trivial for the spin 0 case, but for the spin $\frac{1}{2}$ case leads to a novel derivation of the Dirac equation. Once the equations are obtained, a Hamiltonian formulation is made to give the physical interpretation of the equations. In each case the relativistic generalization results in adding an additional degree of freedom to the nonrelativistic description. This degree of freedom is found to be the electric charge. Associated phenomena such as the nonlocalizability of a particle, "Zitterbewegung" and their relation to the so-called relativistic corrections (e.g., the Darwin term) are discussed. In Sec. 4, the field theoretic interpretation, in which the wave equations give the equations of motion for field operators is connected with the particle interpretation of Sec. 2 and Sec. 3. In particular, the vacuum and one particle state is carefully defined. The wave functions of the single particle theory turn out to be matrix elements between the vacuum and one particle state. With this background, it is then possible to discuss the theory of the neutral particle and, in particular, the Majorana abbreviation and to formulate the theory of the isotopic spin one particle. Although we have considered only spin 0 and spin $\frac{1}{2}$ particles, the methods we have used could be employed to discuss the Proca equation for spin 1.

The coordinates x_k will as usual be x, y, z, as k goes from one to three, $x_4=ict$, while $x_0=ct$. We also use the symbol $\mathbf{x}$ to represent a vector whose components are x_k. Four vectors generally will be indicated by a Greek subscript, e.g., $x_\mu=(x_k,x_4)$. Similar notation is used for the momentum and electromagnetic potentials, $A_k=(A_x,A_y,A_z)$, $A_4=i\phi$, $A_0=\phi$, where A_k is the vector potential and ϕ, the scalar.

2. WAVE MECHANICS OF CHARGED SPIN ZERO PARTICLES

A. The Klein-Gordon Equation[5]

Our general procedure for obtaining the proper relativistic equation describing a particle of a given spin will be to seek the appropriate generalization of the nonrelativistic equation. For the spin zero particle, the nonrelativistic equation for a spin zero particle is the Schroedinger equation,

$$\sum_{k=1}^{3}\frac{1}{2m}\left[\frac{\hbar}{i}\frac{\partial}{\partial x_k}-\frac{e}{c}A_k\right]^2\psi=\left(\frac{\hbar}{i}\frac{\partial}{\partial t}-e\phi\right)\psi, \tag{2.1}$$

where ψ is the Schroedinger wave function and A_k and ϕ are the vector and scalar potential for the electromagnetic field. This equation is obtained from the nonrelativistic Hamiltonian describing the interaction between a charged particle and the electromagnetic field,

$$H(\mathbf{p},\mathbf{r})=1/2m\sum(p_k-(e/c)A_k)^2+e\phi \tag{2.2}$$

by the well-known substitutions derived, for example, by transformation theory from the commutation relations between $\mathbf{p}$ and $\mathbf{r}$,

$$p_k\rightarrow(\hbar/i)(\partial/\partial x_k),\quad H\rightarrow\hbar i(\partial/\partial t). \tag{2.3}$$

To obtain the relativistic generalization of (2.1) we need only write the relativistic energy-momentum relation and make substitutions (2.3). We replace (2.2) by

$$\sum_{\mu=1}^{4}\left(p_\mu-\frac{e}{c}A_\mu\right)^2+(mc)^2=0, \tag{2.4}$$

where

$$\begin{aligned}&p_\mu=(p_k,p_4),\quad p_4=(i/c)H,\\&A_\mu=(A_k,A_4),\quad A_4=i\phi.\end{aligned} \tag{2.5}$$

On inserting substitutions (2.3) (note $p_4\rightarrow(\hbar/i)(\partial/\partial x_4)$) in (2.4) we obtain

$$\left[\sum_\mu[D_\mu^2-\kappa^2]\right]\psi=0, \tag{2.6}$$

where

$$D_\mu=\partial/\partial x_\mu-(ie/\hbar c)A_\mu\quad\text{and}\quad\kappa=mc/\hbar. \tag{2.7}$$

From (2.6) a current four-vector j_μ may be obtained which satisfies the continuity equation,

$$\sum_\mu\partial j_\mu/\partial x_\mu=0, \tag{2.8}$$

where

$$j_\mu=\text{const}[\psi^*D_\mu\psi-(D^\dagger{}_\mu\psi^*)\psi], \tag{2.9}$$

and

$$D^\dagger{}_\mu=\partial/\partial x_\mu+(ie/c\hbar)A_\mu.$$

Equation (2.6) is known as the Klein-Gordon equation. When originally presented, various objections were raised to its use as a particle equation. Firstly, the equation of motion (2.6) involved second as well as first time derivatives. Secondly, the density j_0 is not positive definite; hence, j_μ is not a particle current density. These two apparent difficulties are not unconnected. The presence of the second time derivative indicates that ψ has two degrees of freedom rather than the single degree of freedom of the Schroedinger equation (2.1). Pauli and Weisskopf[6] showed that there is no difficulty of interpretation if the Klein-Gordon equation is regarded as the equation of motion of a field and quantized in the usual fashion. They demonstrated that

[5] The previously mentioned papers by Heitler, Kemmer, and Sakata and Taketani were particularly useful here. The Klein-Gordon equation was proposed by a number of authors: O. Klein, Z. Physik **37**, 895 (1926); V. Fock, Z. Physik **38**, 242 (1926) and **39**, 226 (1926); J. Kudar, Ann. phys. **81**, 632 (1926); W. Gordon, Z. Physik **40**, 117 (1926); and Th. deDonder and H. vanDungen, Compt. rend. (July, 1926).

[6] W. Pauli and V. F. Weisskopf, Helv. Phys. Acta **1**, 709 (1934).

the two degrees of freedom of ψ correspond to two different possible charge states. The current given by (2.9) is then not the particle current but rather the charge current; the fourth component is the charge density. It is clear that the charge density measures the difference between the number of positive and the number of negative particles and in consequence will not be positive-definite. However, it is still possible in the one particle situation that the density is either positive or negative definite depending on the sign of the charge and thus permit a single particle interpretation of the Klein-Gordon equation. This turns out to be the case. It is characteristic of all the relativistic generalizations of particle equations of motion that they will involve both signs of the charge, although the original nonrelativistic equation described only a charge of a given sign.

One of the purposes of the discussion here is to show that a particle interpretation of the Klein-Gordon equation (2.6) is possible if the separation of the two degrees of freedom is made explicit. We expect this form of the equation to be linear in $(\partial/\partial t)$ and so permit the determination of a Hamiltonian for the system.

B. A Hamiltonian Form of the Klein-Gordon Equation

We look for an appropriate Ψ such that Ψ satisfies the Hamiltonian form for the wave equation:

$$H\Psi = i\hbar(\partial\Psi/\partial t). \tag{2.10}$$

To obtain this form it is necessary to resolve ψ into the components representing the two degrees of freedom implied by (2.6). The function Ψ is then a unicolumnar matrix formed from these two components.

The obvious first step in such a development is to introduce $\partial\psi/\partial t$ as an independent component. Let

$$\psi_4 = -\kappa^{-1}D_4\psi. \tag{2.11}$$

Then the Klein-Gordon equation may be written in a form equivalent to (2.6) as follows:

$$\begin{aligned} D_4\psi + \kappa\psi_4 &= 0, \\ \textstyle\sum_k D_k^2\psi - \kappa D_4\psi_4 - \kappa^2\psi &= 0. \end{aligned} \tag{2.12}$$

These equations are already in the Hamiltonian form (2.10), but the combination ψ and ψ_4 does not prove to be convenient because of the asymmetry of (2.12). Accordingly, we introduce the linear combination,

$$\begin{aligned} \psi &= 1/\sqrt{2}(\varphi+\chi), \\ \psi_4 &= 1/\sqrt{2}(\varphi-\chi). \end{aligned} \tag{2.13}$$

The equations for φ and χ are

$$\begin{aligned} D_4\varphi &= (1/2\kappa)\textstyle\sum_k D_k^2(\varphi+\chi) - \kappa\varphi, \\ D_4\chi &= -(1/2\kappa)\textstyle\sum_k D_k^2(\varphi+\chi) + \kappa\chi, \end{aligned} \tag{2.14}$$

which may be written more explicitly as

$$\begin{aligned} i\hbar(\partial\varphi/\partial t) &= (1/2m)(\hbar/i\nabla - e\mathbf{A}/c)^2(\varphi+\chi) \\ &\quad + (e\phi + mc^2)\varphi, \\ i\hbar(\partial\chi/\partial t) &= -(1/2m)(\hbar/i\nabla - e\mathbf{A}/c)^2(\varphi+\chi) \\ &\quad + (e\phi - mc^2)\varphi. \end{aligned} \tag{2.15}$$

For greater compactness we introduce the two-component wave function,

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \tag{2.16}$$

together with the associated Pauli matrices,

$$\begin{aligned} \tau_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \\ \tau_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.17}$$

The Hamiltonian in (2.10) is now seen to be

$$H = (\tau_3 + i\tau_2)(1/2m)(\mathbf{p} - e\mathbf{A}/c)^2 + mc^2\tau_3 + e\phi. \tag{2.18}$$

Similarly the current j_μ may be re-expressed in terms of Ψ and Ψ^*. From (2.8), inserting (2.11) and placing the multiplicative constant equal to ic, we obtain

$$\rho = \varphi^*\varphi - \chi^*\chi = \Psi^*\tau_3\Psi, \tag{2.19}$$

$$\begin{aligned} \mathbf{j} = \hbar/2im[&\Psi^*\tau_3(\tau_3+i\tau_2)\nabla\Psi - (\nabla\Psi^*)\tau_3(\tau_3+i\tau_2)\Psi] \\ &- (e/mc)\mathbf{A}\Psi^*\tau_3(\tau_3+i\tau_2)\Psi. \end{aligned} \tag{2.20}$$

The advantages of using the particular combination φ and χ are: First, the density ρ appears as the difference of two positive definite densities as one would expect in a theory simultaneously describing particles of both signs of charge. Secondly, when φ and χ are used as components of the wave function, the fundamental *charge symmetry* of the relativistic formalism becomes apparent. This is most easily seen from looking at the complex conjugate to (2.15):

$$\begin{aligned} i\hbar\frac{\partial\chi^*}{\partial t} &= \frac{1}{2m}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}A\right)^2(\varphi^*+\chi^*) \\ &\quad + (mc^2 - e\phi)\chi^*, \\ i\hbar\frac{\partial\varphi^*}{\partial t} &= -\frac{1}{2m}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}\right)^2(\varphi^*+\chi^*) \\ &\quad - (mc^2 + e\phi)\varphi^*, \end{aligned} \tag{2.15a}$$

which shows that, if

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

satisfies the equation

$$H(e)\Psi = i\hbar(\partial\Psi/\partial t),$$

there exists a "charge conjugate" wave function

$$\Psi_c = \begin{pmatrix} \chi^* \\ \varphi^* \end{pmatrix}, \tag{2.21}$$

which satisfies the equation $H(-e)\Psi_c = i\hbar(\partial\Psi_c/\partial t)$. $H(-e)$ differs from $H(e)$ by a change of sign of the electric charge. Notice that if Ψ is a positive energy eigenstate of $H(e)$, the associated Ψ_c is a negative energy eigenstate of $H(-e)$, and vice versa. We see also that

$$\rho_c = \Psi_c^* \tau_3 \Psi_c = -\rho, \tag{2.22}$$

and

$$\mathbf{j}_c = \mathbf{j}. \tag{2.23}$$

$\mathbf{j}$ does not change sign because, according to Eq. (2.21), Ψ_c describes a system in which all momenta[7] are reversed in addition. The important point now is that, in the limit of $e \to 0$, the charge conjugate solution Ψ is also a solution of (2.15) so that its two independent solutions are Ψ and Ψ_c. Moreover, we see from (2.9) and (2.22) that, if Ψ is normalized so that the charge,

$$\int \Psi^* \tau_3 \Psi d^3x = \pm 1, \tag{2.24a}$$

the normalization for the charge conjugate solution is of opposite sign, *viz.*,

$$\int \Psi_c^* \tau_3 \Psi_c d^3x = \mp 1. \tag{2.24b}$$

Hence for the no field case there is a one-to-one correspondence between the "positive" and "negative" solutions[8] of the wave equation (2.15), the positive solution, normalized to $+1$ describing a positive charge, the negative a negative charge. This correspondence must survive if an electromagnetic field is introduced.

The main clarification achieved so far is the insight that the increase of the "degrees of freedom" connected with the appearance of a second-order time derivative in the Klein-Gordon equation corresponds to the simultaneous description of a particle of either positive or negative charge; i.e., the value of the charge becomes a degree of freedom of the system. The solution describing a particle of positive charge may be normalized to $+1$; the charge conjugate solution will automatically be normalized to (-1) and thus describe a negative charge.

We may well ask whether there is a relativistic equation describing just one sign of the charge. For free particles it is possible to transform (2.15) into a form in which *two uncoupled* equations appear, one having positive, the other negative solutions only. But as soon as an electromagnetic field is present, this transformation is no longer possible. We discuss in Sec. 4 the physical reason for this and find it in the "polarizability of the vacuum." The charge density ρ (2.20) is, in fact, the actual minus the vacuum charge density, the latter being a polarization density induced by the external electromagnetic field. The necessity of introducing a field-dependent definition of vacuum is responsible for the general inseparability of the system (2.15) into an equation for positive and an equation for negative charge.

C. Free-Particle Solutions

Before discussing the physical content of the above development in more detail, it will be useful to have available the free-particle solutions of the Schroedinger equation with Hamiltonian of (2.18) ($A_\mu = 0$). We insert

$$\Psi = \begin{pmatrix} \varphi_0(\mathbf{p}) \\ \chi_0(\mathbf{p}) \end{pmatrix} e^{i/\hbar(\mathbf{p}\cdot\mathbf{x} - Et)}$$
$$\Psi = \Psi_0(\mathbf{p}) e^{i/\hbar(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{2.25}$$

into the Schroedinger equation (2.15) and obtain

$$(E - mc^2)\varphi_0 = (p^2/2m)(\varphi_0 + \chi_0),$$
$$(E + mc^2)\chi_0 = -(p^2/2m)(\varphi_0 + \chi_0).$$

There are two possible values for E,

$$E = \pm E_p; \quad E_p = [(cp)^2 + (mc^2)^2]^{\frac{1}{2}}. \tag{2.26}$$

The associated solutions are

$$E = E_p$$
$$\Psi_0^{(+)}(\mathbf{p}): \quad \left.\begin{aligned} \varphi_0^{(+)} &= \frac{E_p + mc^2}{2(mc^2 E_p)^{\frac{1}{2}}} \\ \chi_0^{(+)} &= \frac{mc^2 - E_p}{2(mc^2 E_p)^{\frac{1}{2}}} \end{aligned}\right\} \varphi_0^2 - \chi_0^2 = 1,$$
$$E = -E_p \tag{2.27}$$
$$\Psi_0^{(-)}(\mathbf{p}): \quad \left.\begin{aligned} \varphi_0^{(-)} &= \frac{mc^2 - E_p}{2(mc^2 E_p)^{\frac{1}{2}}} \\ \chi_0^{(-)} &= \frac{E_p + mc^2}{2(mc^2 E_p)^{\frac{1}{2}}} \end{aligned}\right\} \varphi_0^2 - \chi_0^2 = -1.$$

As expected, the "negative" solution ($E = -E_p$) is the solution charge conjugate to the "positive" solution ($E = E_p$) so that if the positive solution represents a particle of positive charge the $(-)$ solution represents one of negative charge. In the nonrelativistic limit $\varphi_0^{(+)} \sim 1$, $\chi_0^{(+)} \sim (\mathbf{p}/mc)^2$. The same holds for $\chi_0^{(-)}$ and $\varphi_0^{(-)}$. We also observe the orthogonality relations,

$$\Psi_0^{(+)}(\mathbf{p})\tau_3\Psi_0^{(-)}(\mathbf{p}) = 0 = \Psi_0^{(-)}(\mathbf{p})\tau_3\Psi_0^{(+)}(\mathbf{p}). \tag{2.28}$$

However, $\Psi_0^{(+)}$ and $\Psi_0^{(-)}$ are not eigenvectors of the operator τ_3.

The nonrelativistic limit of (2.15) is of interest. For weak external fields and nonrelativistic velocities χ will be negligible compared to φ, since the ratio is of order $(v/c)^2$. Dropping the small component χ in (2.15) and eliminating the mc^2 term by putting $\varphi(x,t) = \varphi_{NR}$

[7] For a detailed demonstration use the free-particle solutions of Sec. 2C, Eqs. (2.25) and (2.27).

[8] This terminology is an abbreviation for the solutions with positive and negative normalizations. In the future the phrases positive and negative charge solutions and positive and negative solutions will be employed interchangeably.

$\times\exp(imc^2t/\hbar)$ yields the usual nonrelativistic Schroedinger equation for a $+e$ charge,

$$i\hbar\frac{\partial\varphi_{NR}}{\partial t}=\frac{1}{2m}[(\hbar/i)\nabla-(e/c)\mathbf{A}]^2\varphi_{NR}+e\phi\varphi_{NR}. \quad (2.29)$$

An analogous equation for $(\varphi_c)_{NR}$ may be obtained, only the sign of e is reversed.

D. Operators and Expectation Values

The definition of the expectation value of an operator Ω is, as expected from the expression for the normalization $\Psi^*\tau_3\Psi$, given by

$$\langle\Omega\rangle=\int\Psi^*\tau_3\Omega\Psi d^3x. \quad (2.30)$$

The principal justification of this definition lies in the result derived below that it yields operator equations of motion and other expectation values in complete accord with the correspondence principle.

Definition (2.30) implies a particular definition of Hermitivity and unitarity which differs from the nonrelativistic definitions. We define the Hermitian adjoint $\Omega^\dagger$ of an operator Ω by the equation

$$\int\Psi_a{}^*\tau_3\Omega\Psi_b d^3x=\int(\Omega^\dagger\Psi_a)^*\tau_3\Psi_b d^3x. \quad (2.31)$$

This is not the usual definition of adjoint $\bar{\Omega}=(\Omega^T)^*$; i.e., transpose and complex conjugate used in nonrelativistic theory. The two are related by the equation,

$$\Omega^\dagger=\tau_3\bar{\Omega}\tau_3. \quad (2.32)$$

An operator is Hermitian and will have real expectation values if

$$\Omega=\Omega^\dagger. \quad (2.33)$$

Unitary operators are defined so as to preserve the *charge* of the state. If

$$\Psi'=S\Psi,$$

then

$$\int\Psi'^*\tau_3\Psi' d^3x=\int(S\Psi)^*\tau_3(S\Psi)d^3x=\int\Psi^*\tau_3S^\dagger S\Psi d^3x.$$

Hence, for unitarity

$$S^\dagger S=1 \quad \text{or} \quad S^{-1}=S^\dagger. \quad (2.34)$$

From the requirement that matrix elements be preserved under transformation S, it follows that the transformed operator Ω' is

$$\Omega'=S^{-1}\Omega S. \quad (2.35)$$

This result is important since it indicates that the behavior of operators under various transformations is identical to that obtained in nonrelativistic theory. Moreover, since the usual geometrical transformations of inversion and rotation are independent of τ_i, it follows that they are unitary as defined by (2.32) and (2.34).

The Hamiltonian (2.18) is Hermitian, and, since

$$\tau_3{}^\dagger=\tau_3, \quad \tau_2{}^\dagger=\tau_3\tau_2\tau_3=-\tau_2,$$

it follows that

$$(\tau_3+i\tau_2)^\dagger=\tau_3+i\tau_2.$$

This result has many immediate consequences. The time dependence of a wave function is as usual given by

$$\Psi(t)=e^{-iHt/\hbar}\Psi(0).$$

Since H is Hermitian, the transformation

$$S=e^{-iHt/\hbar}$$

is unitary. The time dependence of any operator is, therefore, given by (2.34) as

$$\Omega(t)=e^{iHt/\hbar}\Omega(0)e^{-iHt/\hbar}. \quad (2.36)$$

It follows that

$$d\Omega/dt=i/\hbar[H,\Omega]=i/\hbar[H\Omega-\Omega H]. \quad (2.37)$$

By taking the expectation value of both sides of (2.37), one obtains an important corollary,

$$\langle d\Omega/dt\rangle=(d/dt)\langle\Omega\rangle. \quad (2.38)$$

We now see that usual definition of "constant of motion" holds. The expectation value of an operator Ω is a constant of the motion if Ω commutes with H. Moreover, any operator which leaves H invariant according to (2.35) commutes with H. Hence, such operators as those of rotation and reflection, orbital angular momentum and parity, respectively, will have expectation values which are constants of the motion. Another important result is that the expectation value of the kinetic energy operator is positive definite since

$$\left\langle(\tau_3+i\tau_2)\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2\right\rangle$$

$$=\int\Psi^*\tau_3(\tau_3+i\tau_2)\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2\Psi d^3x$$

$$=\int(\varphi^*+\chi^*)\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2(\varphi+\chi)d^3x\geq 0.$$

Hence the energies associated with the free-particle solutions are positive and indeed equal to E_p.

So far the discussion has in nearly every respect paralleled the discussion for the nonrelativistic situation. The main difference is in the normalization condition and the corresponding change in the definition of the Hermitian adjoint to an operator. The equations of motion of operators and the significance of H as the time displacement operator remain the same as in the

nonrelativistic situation. However, there are important differences from the Schroedinger case beside the normalization to ± 1 (see (2.26) and (2.27)).

Perhaps the most obvious difference between the two descriptions is embodied in the quite different roles played by the eigenvalues of a given operator. In Schroedinger theory, the eigenvalues are the expectation values of the operator for the corresponding eigenstates. For the spin zero equation this is no longer true. For example, the Hamiltonian H of (2.18) has, in the free-particle case, eigenvalues of $\pm E_p$, but expectation values of E_p only. Moreover, this value of E_p is sharp in the sense that the fluctuation vanishes. For the eigenstates of H,

$$\langle [H-\langle H\rangle]^2\rangle = 0.$$

Another striking example is given by the velocity operator. From (2.37) it follows that the canonical equations of motion are valid as operator equations so that for the free-particle case

$$\frac{\partial H}{\partial p_k} = v_k = \frac{p_k}{m}(\tau_3 + i\tau_2). \tag{2.39}$$

The expectation value for a plane wave state is

$$\langle v_k\rangle = (p_k/m)\int \Psi^* \tau_3(\tau_3+i\tau_2)\Psi d^3x$$

$$= (p_k/m)\,|\varphi_0+\chi_0|^2 = c^2 p_k/E_p,$$

a not unexpected result. {A direct evaluation of the expectation value of j_k [Eq. (2.20)] gives the result $\langle j_k\rangle = \langle v_k\rangle$ for a plane wave, as it should.} On the other hand, the eigenvalues of the operator v_k are zero since $(\tau_3+i\tau_2)^2=0$. This is not inconsistent with the result obtained for the expectation value of v_k because the eigenstates of v_k do not form a complete set contrary to the usual experience in nonrelativistic theory where the eigenstates of a Hermitian operator do generally form a complete set. Another way of expressing this result is to say that v_k is not an observable in relativistic spin zero theory, completeness of eigenstates being necessary for application of statistical theory. In any event we see from this example that eigenvalues in this theory no longer have the direct physical significance they have in nonrelativistic theory.

Another important difference from nonrelativistic theory is in the statistical interpretation. To discuss it we need a complete set of states. These are provided by the eigenstates of the Hamiltonian, $\Psi_n^{(+)}$ and $\Psi_n^{(-)}$ where n numbers the eigenstates and the superscript gives the positive or negative character of the states, respectively, as in (2.27). For later convenience we introduce the notation:

$$\Psi_{n,1} \equiv \Psi_n^{(+)}; \quad \Psi_{n,-1} \equiv \Psi_n^{(-)}.$$

Hence,

$$H\Psi_{n,\sigma} = E_{n\sigma}\Psi_{n,\sigma} \quad (\sigma = \pm 1,\ n = 1, 2, \cdots),$$

with the orthogonality relation

$$\int \Psi_{n\sigma}^* \tau_3 \Psi_{n'\sigma'} d^3x = \sigma\delta_{\sigma\sigma'}\delta_{nn'}. \tag{2.40}$$

These states are complete since they form all the solutions of the original Klein-Gordon equation.

We therefore may expand any state Ψ representing a particle with a given charge ± 1 in terms of $\Psi_{n\sigma}$,

$$\Psi = \sum_{n\sigma} a_{n\sigma}\Psi_{n\sigma}; \quad a_{n\sigma} = \sigma\int d^3x \Psi_{n\sigma}^* \tau_3 \Psi. \tag{2.41}$$

If we now ask for the normalization integral, we obtain by (2.24)

$$\int \Psi^* \tau_3 \Psi d^3x = \sum [\,|a_{n,1}|^2 - |a_{n,-1}|^2] = \pm 1, \tag{2.42}$$

depending on whether Ψ is a positive or negative state. The interpretation of this equation is as follows. The integral $|a_{n,1}|^2$ gives the relative amount of *positive* charge which is spatially distributed according to eigenstate Ψ_{n1}, while $|a_{n,-1}|^2$ gives the relative amount of *negative* charge distributed according to state $\Psi_{n,-1}$. The total amount of charge is ± 1. A similar description is equally valid for the Hamiltonian operator:

$$\langle H\rangle = \int \Psi^* \tau_3 H \Psi d^3x = \sum_{n\sigma} \sigma E_{n\sigma} \left| \int \Psi_{n\sigma}^* \tau_3 \Psi d^3x \right|^2.$$

For sufficiently weak interactions there will be a definite sign of the energy associated with each value of σ so that $\sigma E_{n\sigma} = |E_{n\sigma}|$ and

$$\langle H\rangle = \sum_{n,\sigma} |E_{n\sigma}| \left| \int \Psi_{n\sigma}^* \tau_3 \Psi d^3x \right|^2. \tag{2.43}$$

This equation states that the expectation value for the energy is the sum of the energies for each state n, charge σ weighted by the amount of charge in each state. It is clear that a similar interpretation will be possible for any operator which commutes with H.

To summarize, it has been shown that the Klein-Gordon equation admits of a single particle interpretation in which the particle necessarily possesses a charge degree of freedom. To every solution with a given sign of charge there is a corresponding charge conjugate solution with the opposite sign [see (2.24)] and both of these solutions are needed to give a complete description of possible solutions of the Klein-Gordon equation representing a particle of a given sign of charge [see (2.41)]. We also find that it is possible to set up the general transformation theory in which H has its usual significance of the infinitesimal time displacement oper-

ator, (2.37) and in which parity and angular momentum play their customary roles. Physically we expect this description to break down if the external electromagnetic fields are sufficiently strong so that pairs can be produced and, of course, a single particle theory would then be inadequate. This appears from the preceding analysis in that it is not possible to make a clean separation of positive and negative solutions if the external fields are too strong. We would like to emphasize, however (proved in Sec. 4) that the effects of virtual pair production are completely taken into account. Indeed, some of the effects of these virtual processes are revealed in the phenomenon of Zitterbewegung as shown in the following section.

E. Free-Particle Representation

For a more detailed discussion of the interpretation of the wave equation, particularly its nonrelativistic limit, and also the connection with the formalism of quantum field theory, it is more convenient to have the wave equation in a representation in which the free particle Hamiltonian is diagonal rather than (2.17), in which the position operator $\mathbf{x}$ is diagonal. In such a representation, the wave equation for a free particle separates into two uncoupled equations, one describing a particle of positive charge, the second a particle of negative charge. If an external electromagnetic field $(\mathbf{A},\phi)$ is present, the equation will be shown not to separate any more. In Sec. 4 we show that this coupling between the two equations is an effect of the polarization of the vacuum.

Since the free particle Hamiltonian can only be diagonal in momentum space, we first write Eq. (2.15) in p space. We just have to make the replacements,

$$\mathbf{x}\rightarrow i\hbar\nabla_p,$$
$$\hbar/i\nabla\rightarrow\mathbf{p},$$

into (2.15) to have the transformed equation. The transformed equation therefore reads

$$i\hbar\frac{\partial\Psi(\mathbf{p},t)}{\partial t}=\left\{\frac{1}{2m}\left[\mathbf{p}-\frac{e}{c}A(i\hbar\nabla_p)\right]^2(\tau_3+i\tau_2)+mc^2\tau_3+e\phi(i\hbar\nabla_p)\right\}\Psi(\mathbf{p},t). \quad (2.44)$$

The operator $\phi(i\hbar\nabla_p)$ is most easily defined in terms of its Fourier transform $\Phi(\mathbf{q})$ where

$$\phi(\mathbf{x})=\int e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\Phi(\mathbf{q})d^3q. \quad (2.45)$$

Then

$$\phi(i\hbar\nabla_p)=\int\exp-(\mathbf{q}\cdot\nabla_p)\Phi(\mathbf{q})d^3q.$$

Moreover, from the properties of the exponential operator,

$$\exp(-\mathbf{q}\cdot\nabla_p)f(\mathbf{p})=f(\mathbf{p}-\mathbf{q}), \quad (2.46)$$

so that

$$\phi(i\hbar\nabla_p)f(\mathbf{p})=\int\Phi(\mathbf{q})f(\mathbf{p}-\mathbf{q})d^3q$$

or

$$\phi(i\hbar\nabla_p)f(\mathbf{p})=\int\Phi(\mathbf{p}-\mathbf{q})f(\mathbf{q})d^3q,$$

a familiar result. The free-particle Hamiltonian is

$$H_0=(\tau_3+i\tau_2)(p^2/2m)+\tau_3mc^2, \quad (2.47)$$

and has (Sec. 2C) the eigenvalues $\pm E_p$ and eigenvectors $\Psi^{(+)}(\mathbf{p})$, $\Psi^{(-)}(\mathbf{p})$ given by (2.26).

The transformation to the free-particle representation can now be made specific. Let us introduce the basis-vectors

$$\eta^{(+)}=\begin{pmatrix}1\\0\end{pmatrix},\quad \eta^{(-)}=\begin{pmatrix}0\\1\end{pmatrix}, \quad (2.48)$$

in terms of which we may write

$$\Psi(\mathbf{p},t)=\begin{pmatrix}\varphi(\mathbf{p},t)\\ \chi(\mathbf{p},t)\end{pmatrix}=\varphi\eta^{(+)}+\chi\eta^{(-)}. \quad (2.49)$$

In the free-particle representation, we replace φ and χ by the amplitudes u and v with respect to new basis vectors $\Psi_0^{(+)}$ and $\Psi_0^{(-)}$ as given in (2.27). That is, we write, instead of (2.49),

$$\Psi(\mathbf{p},t)=u(\mathbf{p},t)\Psi_0^{(+)}(\mathbf{p})+v(\mathbf{p},t)\Psi_0^{(-)}(\mathbf{p}). \quad (2.50)$$

The choice of this new basis is simply a "rotation" in τ space; both $\eta^{(\pm)}$ and $\Psi_0^{(\pm)}$ form orthonormal sets and are hence connected by a unitary transformation.

Now a comparison of (2.49) and (2.50), both being expressions for the *same* two-component wave function $\Psi(\mathbf{p},t)$, leads to

$$\Psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}\varphi_0^{(+)} & \varphi_0^{(-)}\\ \chi_0^{(+)} & \chi_0^{(-)}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=U\begin{pmatrix}u\\ v\end{pmatrix}=U\Psi^{\mp}, \quad (2.51)$$

where

$$\Psi^{\mp}=\begin{pmatrix}u\\ v\end{pmatrix}$$

is the wave function in the free particle representation.[9] Hence, the transformation U which connects with the transformed $\Psi^{\mp}$ is given by a matrix whose columns are the eigenvectors $\Psi_0^{(\pm)}(\mathbf{p})$. By (2.27),

$$U=\frac{1}{2(mc^2E_p)^{\frac{1}{2}}}\begin{pmatrix}E_p+mc^2 & mc^2-E_p\\ mc^2-E_p & E_p+mc^2\end{pmatrix},$$

[9] Throughout this paper we use the $\mp$ to indicate operators and states in the free-particle representation.

or

$$U=[1/2(mc^2E_p)^{\frac{1}{2}}][(E_p+mc^2)-\tau_1(E_p-mc^2)], \quad (2.52a)$$

and

$$U^{-1}=[1/2(mc^2E_p)^{\frac{1}{2}}] \times[(E_p+mc^2)+\tau_1(E_p-mc^2)]. \quad (2.52b)$$

It is obvious that

$$U^{-1}(p)H_0(p)U(p)=\tau_3E_p. \quad (2.53)$$

Inserting $\Psi=U\Psi^+$ into wave equation (2.44) we obtain the equation for Ψ^+,

$$i\hbar\frac{\partial\Psi^+}{\partial t}=\tau_3E_p\Psi^+ +U^{-1}(p)\left\{-\frac{e}{mc}[\mathbf{p}\cdot\mathbf{A}(i\hbar\nabla_p)+\mathbf{A}(i\hbar\nabla_p)\cdot\mathbf{p}] +\frac{e^2}{2mc^2}A^2(i\hbar\nabla_p)+e\phi(i\hbar\nabla_p)\right\} \times U(p)\Psi^+(p)=0. \quad (2.54)$$

The last term in this equation is easily written in a more explicit fashion by using the two steps illustrated in (2.45) and (2.46). Using the notation,

$$\phi(\mathbf{q}), \quad \mathbf{A}(\mathbf{q}), \quad A^2(\mathbf{q}),$$

for the Fourier transform of the corresponding space functions, we obtain terms of type

$$eU^{-1}(p)\int d^3q\phi(\mathbf{p}-\mathbf{q})U(q)\Psi^+(\mathbf{q}).$$

Now $U^{-1}(p)U(p)=1$ but

$$U^{-1}(p)U(q)=\frac{E_p+E_q}{2(E_pE_q)^{\frac{1}{2}}}+\tau_1\frac{E_p-E_q}{2(E_pE_q)^{\frac{1}{2}}}.$$

Using this and similar relations, (2.54) reads, in detail,

$$i\hbar\frac{\partial u}{\partial t}=E_pu(\mathbf{p},t)+\int d^3qe\phi(\mathbf{p}-\mathbf{q}) \times\left\{\frac{E_p+E_q}{2(E_pE_q)^{\frac{1}{2}}}u(\mathbf{q},t)+\frac{E_p-E_q}{2(E_pE_q)^{\frac{1}{2}}}v(\mathbf{q},t)\right\} +\int d^3q\frac{mc^2}{2(E_pE_q)^{\frac{1}{2}}}\left\{-\frac{e}{mc}\mathbf{q}\cdot\mathbf{A}(\mathbf{p}-\mathbf{q}) +\frac{e^2}{2mc^2}A^2(\mathbf{p}-\mathbf{q})\right\}(u+v), \quad (2.55a)$$

$$i\hbar\frac{\partial v}{\partial t}=-E_pv(\mathbf{p},t)+\int d^3qe\phi(\mathbf{p}-\mathbf{q}) \times\left\{\frac{E_p+E_q}{2(E_pE_q)^{\frac{1}{2}}}v(\mathbf{q},t)+\frac{E_p-E_q}{2(E_pE_q)^{\frac{1}{2}}}u(\mathbf{q},t)\right\} +\int d^3q\frac{mc^2}{2(E_pE_q)^{\frac{1}{2}}}\left\{-\frac{e}{mc}\mathbf{q}\cdot\mathbf{A}(\mathbf{p}-\mathbf{q}) +\frac{e^2}{2mc^2}A^2(\mathbf{p}-\mathbf{q})\right\}(u+v). \quad (2.55b)$$

We now investigate the consequences of these equations in the weak field (i.e., fields with small high momentum Fourier amplitudes) limit. Such weak fields couple u and v only very weakly so that a useful approximation is obtained, say for the positive solution, by dropping v in the equation for u.

F. Discussion of Weak Field Limit

We discuss separately the effect of an electric and a magnetic field.

(1) We consider a weak electric field, e.g., that given by a light nucleus ($Z\ll137$). Equation (2.55) for u becomes

$$i\hbar\frac{\partial u}{\partial t}=E_pu+\int d^3qe\phi(\mathbf{p}-\mathbf{q}) \times\left[\frac{E_p+E_q}{2(E_pE_q)^{\frac{1}{2}}}u(\mathbf{q})+\frac{E_p-E_q}{2(E_pE_q)^{\frac{1}{2}}}v(\mathbf{q})\right]. \quad (2.56)$$

In the equation for v we may write approximately

$$v\simeq\frac{1}{E+E_p}\int d^3qe\phi(\mathbf{p}-\mathbf{q})\frac{p^2-q^2}{(2mc^2)^2}u(\mathbf{q}), \quad (2.57)$$

where we have replaced $i\hbar\partial/\partial t$ by Ev. Inserting this result into (2.56), the correction to u from coupling to v is quadratic in the electrostatic potential. Hence in case the field is sufficiently weak, it is negligible compared to the u term on the right-hand side of (2.56). For example, for the case of a Coulomb field $\phi(\mathbf{x})=Ze/r$ and using the exact nonrelativistic solution on the right-hand side of (2.57) we find

$$v/u\simeq\frac{1}{8}\left(\frac{Ze^2}{\hbar c}\right)^4,$$

so that for small enough Z we may drop v entirely in (2.56). Dropping v, we get, from (2.55),

$$i\hbar\frac{\partial u}{\partial t}=E_pu+\int d^3qe\phi(\mathbf{p}-\mathbf{q})\frac{E_p+E_q}{2(E_pE_q)^{\frac{1}{2}}}u(\mathbf{q}). \quad (2.58)$$

In this form, and expansion of E_p and E_q,

$$E_p=mc^2+(p^2/2m)-(p^4/8m^3c^2),$$

gives the corrections to the low energy limit. For charged scalar particles the first relativistic corrections to the Coulomb potential are of order $(p/mc)^4$; in sharp contrast to the spin $\frac{1}{2}$ case, where they appear in $(p/mc)^2$ (spin-orbit and Darwin terms). Hence, in our case, to order $(v/c)^2$, the equation for a spin 0 particle of given charge reads simply (after taking out the time dependence associated with the rest mass),

$$i\hbar\frac{\partial u}{\partial t}=\frac{p^2}{2m}\left(1-\frac{p^2}{4m^2c^2}\right)u+\int d^3q e\phi(\mathbf{p}-\mathbf{q})u(\mathbf{q}).$$

The only $(v/c)^2$ correction is the relativistic increase in mass.

(2) Zeeman effect of a π-mesic atom. A weak magnetic field adds to (2.56) the coupling term

$$-\frac{e}{mc}\int d^3q\frac{mc^2}{(E_pE_q)^{\frac{1}{2}}}\mathbf{q}\cdot\mathbf{A}(\mathbf{p}-\mathbf{q})u(\mathbf{q}),$$

where we use a gauge in which $\mathrm{div}\mathbf{A}=0$, or $(\mathbf{p}-\mathbf{q})\cdot\mathbf{A}(\mathbf{p}-\mathbf{q})=0$.

For a homogeneous magnetic field we have from, $\mathbf{A}(\mathbf{x})=\frac{1}{2}(\mathbf{H}\times\mathbf{r})$,

$$\mathbf{A}(\mathbf{p}-\mathbf{q})=\frac{i\hbar}{2}\mathbf{H}\times\nabla_p\delta(\mathbf{p}-\mathbf{q}).$$

The coupling term then becomes

$$-(e/2mc)(mc^2/E_p)\mathbf{H}\cdot(i\hbar\nabla_p\times\mathbf{p})u(\mathbf{p}),$$

or

$$-(e/2mc)(mc^2/E_p)\mathbf{H}\cdot\mathbf{L}u(\mathbf{p}).$$

Therefore, the magnetic moment operator $\boldsymbol{\mu}$ is

$$\boldsymbol{\mu}=(e/2mc)(mc^2/E_p)\mathbf{L}. \tag{2.59}$$

The relativistic effects, therefore, *reduce* the Zeeman splitting in the π-mesic atom.

G. The Zitterbewegung

The discussion in this section provides a better understanding of the origin of the correction terms to the Coulomb interaction in (2.56). We had already observed that the positive states alone do not satisfy a completeness relation. The narrowest packet that can be built up of positive states alone has a width of order $\hbar/mc$. To construct a δ function, negative and positive states must contribute with equal weight.

It follows that eigenstates of the position operator $\mathbf{x}$ [that is, δ function $\delta(\mathbf{x}-\mathbf{x}_0)$] necessarily contain positive and negative components. Defining even (odd) operators as having nonzero matrix elements only between states of equal (opposite) charge,[10] it follows that the position operator is *not* an even operator. $H_0(\mathbf{p})=(\tau_3+i\tau_2)(p^2/2m)+\tau_3mc^2$ is an example of purely *even* operator, since

$$H_0(\mathbf{p})\Psi_0^{(\pm)}(\mathbf{p})=\pm E_p\Psi_0^{(\pm)}(\mathbf{p}),$$

and τ_1 is an example of an *odd* operator, since

$$\tau_1\Psi_0^{(\pm)}(\mathbf{p})=\Psi_0^{(\mp)}(\mathbf{p}),$$

as one can easily verify.

Any operator may be decomposed uniquely into an even and an odd part; in particular,

$$\mathbf{x}=i\hbar\nabla_p=\mathbf{x}^{(+)}+\mathbf{x}^{(-)}.$$

We obtain now this decomposition of $\mathbf{x}$. The discussion of the position operator is best carried out in momentum space, due to the close connection of evenness or oddness with the free-particle representation. Since in this representation, positive and negative states are of type

$$\begin{pmatrix}u(\mathbf{p})\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0\\v(\mathbf{p})\end{pmatrix},$$

respectively, an even operator $\Omega^{(+)}$ must, in this representation, be a diagonal matrix,

$$(\Omega^{\pm})^{(+)}=\begin{pmatrix}\omega_1&0\\0&\omega_2\end{pmatrix},$$

whereas, an odd operator $\Omega^{(-)}$ must be nondiagonal

$$(\Omega^{\pm})^{(-)}=\begin{pmatrix}0&\omega_{12}\\\omega_{21}&0\end{pmatrix}.$$

It is easy to write the position operator in this representation. We have

$$\mathbf{x}^{\pm}=(i\hbar\nabla_p)^{\pm}=U^{-1}(\mathbf{p})(i\hbar\nabla_p)U(\mathbf{p}), \tag{2.60}$$

the matrix U being given by (2.52). Now (2.60) may be rewritten as

$$\begin{aligned}\mathbf{x}^{\pm}&=i\hbar\nabla_p+U^{-1}(\mathbf{p})[i\hbar\nabla_p,U(\mathbf{p})]\\&=i\hbar(\nabla_p-c^2\mathbf{p}/2E_p^2\tau_1),\end{aligned} \tag{2.61}$$

as may be found with (2.52).

The important result is that in the free particle representation $i\hbar\nabla_p$ is the even part of $\mathbf{x}^{\pm}$, an operator, for which the name of *mean position operator* has been proposed.[4] The time derivative of the mean position operator defines the mean velocity. For a free particle we have

$$\begin{aligned}\frac{d}{dt}(\mathbf{x}^{\pm})^{(+)}&=\frac{i}{\hbar}[H_0^{\pm},(\mathbf{x}^{\pm})^{(+)}]\\&=\frac{i}{\hbar}[\tau_3E_p,i\hbar\nabla_p]=\tau_3\frac{\partial E_p}{\partial p}=\tau_3c^2p/E_p.\end{aligned} \tag{2.62}$$

[10] An equivalent definition is: Even operators are diagonal matrices in the free-particle representation; odd operators are completely nondiagonal.

This latter property justifies the terminology since the time derivative of $(\mathbf{x}^{\pm})^{(+)}$ gives the correct value of the steady motion of the particle.

If we Fourier transform the wave equation in the free-particle representation back into x space, we get a space-time description in terms of the mean position of the particle. The odd part $(\mathbf{x}^{\pm})^{(-)}$ is clearly an integral operator in x space,

$$(\mathbf{x}^{\pm})^{(-)}\Psi^{\pm}(\mathbf{x},t)=\int d^3x'\mathbf{X}^{(-)}(\mathbf{x}-\mathbf{x}')\Psi^{\pm}(\mathbf{x}',t).$$

The kernel $\mathbf{X}^{(-)}(x-x')$ is given by

$$\mathbf{X}^{(-)}(\mathbf{x}-\mathbf{x}')=-\frac{i\hbar\tau_1}{2(2\pi\hbar)^3}\int d^3p\frac{\mathbf{p}}{p^2+(mc)^2}\times\exp\left[\frac{i\mathbf{p}}{\hbar}\cdot(\mathbf{x}-\mathbf{x}')\right],$$

or

$$\mathbf{X}^{(-)}(\mathbf{x}-\mathbf{x}')=-\frac{\tau_1}{2}\nabla_x\left[\frac{\exp[-(mc/\hbar)|\mathbf{x}-\mathbf{x}'|]}{4\pi|\mathbf{x}-\mathbf{x}'|}\right]. \quad (2.63)$$

From (2.52), $(\mathbf{x}^{\pm})^{(-)}=\mathbf{x}^{(-)}$, so in the original x representation the mean position operator

$$\mathbf{x}^{(+)}=\mathbf{x}-\mathbf{x}^{(-)}$$

is an integral operator with the kernel,

$$\mathbf{X}^{(+)}(\mathbf{x}-\mathbf{x}')=\mathbf{x}\delta(\mathbf{x}-\mathbf{x}')+\frac{\tau_1}{2}\nabla_x\left[\frac{\exp[-(mc/\hbar)|\mathbf{x}-\mathbf{x}'|]}{4\pi|\mathbf{x}-\mathbf{x}'|}\right]. \quad (2.64)$$

It is of interest to construct the *eigenfunction* of $\mathbf{x}^{(+)}$ in the x representation.[11] This is easiest again in the free-particle representation, where $(\mathbf{x}^{\pm})^{(+)}$ is $i\hbar\nabla_p$, so that an eigenfunction with eigenvalue $\mathbf{x}_0$ of $(\mathbf{x}^{\pm})^{(+)}$ has the form at the time $t=0$, the Fourier amplitude:

$$\Psi^{\pm}(\mathbf{x}_0|\mathbf{p})=\exp(i/\hbar\mathbf{p}\cdot\mathbf{x}_0)\begin{pmatrix}1\\0\end{pmatrix}.$$

We transform this into the x representation

$$\Psi(\mathbf{x}_0|\mathbf{x})=1/(2\pi\hbar)^3\int d^3p\,\exp[i/\hbar\mathbf{p}\cdot(\mathbf{x}_0-\mathbf{x})]U(\mathbf{p})\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\frac{1}{(2\pi\hbar)^3}\int d^3p\,\exp[i/\hbar\mathbf{p}\cdot(\mathbf{x}_0-\mathbf{x})]\times\begin{pmatrix}(E_p+mc^2)/2(mc^2E_p)^{\frac{1}{2}}\\(mc^2-E_p)/2(mc^2E_p)^{\frac{1}{2}}\end{pmatrix}. \quad (2.65)$$

The integrations may be performed, giving

$$\Psi(\mathbf{x}_0|\mathbf{x})=\frac{1}{4\pi^2}\left(\frac{mc}{\hbar}\right)^3\frac{1}{s}\frac{d}{ds}\begin{pmatrix}A-B\\-A-B\end{pmatrix},$$

$$A(s)=-\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{\delta(s)}{s^{\frac{1}{2}}}+\frac{\pi^{\frac{1}{2}}}{4\Gamma(\frac{3}{4})}\frac{K_{3/4}(s)}{s^{3/4}}, \quad (2.66)$$

$$B(s)=\frac{\pi^{\frac{1}{2}}2^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}\frac{K_{1/4}(s)}{s^{\frac{1}{4}}};\quad s=\frac{mc}{\hbar}|\mathbf{x}-\mathbf{x}_0|.$$

Compare this result to the eigenfunction of $\mathbf{x}$, $\delta(\mathbf{x}-\mathbf{x}_0)$. The most noteworthy change is that the eigenfunction for $\mathbf{x}^{(+)}$ is not localized, i.e., it has a value different from zero for points where $\mathbf{x}\neq\mathbf{x}_0$. This extension in space covers a radius of the order of $\hbar/mc$ since for large values of s, $A\sim e^{-s}/s^{5/4}$, $B\sim e^{-s}/s^{3/4}$. In other words, the eigenstates of the mean position operator are packets of width $\hbar/mc$. These are the narrowest possible packets, whose space time development follows the nonrelativistic pattern: A propagation with group velocity $v_g=(pc/E_p)_{Av}$, resulting from a displacement of the interference maximum of the various positive Fourier components.

Any sharper localization of a packet brings in negative components, and with this a new element into the space-time behavior: the Schroedinger "Zitterbewegung" (trembling motion). Indeed, since positive and negative parts have a time dependence given by

$$\exp(\mp iE_pt/\hbar),$$

respectively, the density $\rho=\Psi^*\tau_3\Psi$ contains now interference terms which shift the charge around with a frequency of, at least, $2mc^2/\hbar$. This trembling motion is excited whenever the particle is subject to a field of force whose potential varies significantly over a distance of order $\hbar/mc$; the Coulomb potential near a nucleus is an example.

Although we are going to discuss this effect quantitatively, we may well pursue the sketch a little further. A positive packet, subject to such a field, will in first approximation feel the average (taken over a distance $\hbar/mc$) effect of that field. The excitation of negative components appears in next order. As already mentioned, the appearance of these components is best viewed as a polarization effect and is discussed as such later. As a virtual process (which might be called "virtual Zitterbewegung"), negative components also modify the coupling of the positive amplitude with the field. This latter process gives correction terms of order $(p/mc)^4$ for the spin zero particle, and term of order $(p/mc)^2$ for spin $\frac{1}{2}$ particles.

For a more quantitative discussion, we first state again that each operator $\Omega(\mathbf{x})$ (for instance, an external field) appears in the free-particle representation as

$$\Omega^{\pm}=U^{-1}(\mathbf{p})\Omega(i\hbar\nabla_p)U(\mathbf{p}),$$

[11] T. D. Newton and E. P. Wigner, Revs. Modern Phys. **21**, 400 (1949).

which in turn may be written

$$\Omega^{\pm}=\Omega(U^{-1}i\hbar\nabla_p U)=\Omega(\mathbf{x}^{\pm}). \tag{2.67}$$

We may then use (2.61) and obtain

$$\Omega^{\pm}=\Omega[i\hbar\nabla_p-(i\hbar/2)\tau_1(c^2\mathbf{p}/E_p^2)]. \tag{2.68}$$

In particular, we apply (2.68) to the case of an electrostatic potential

$$\phi(i\hbar\nabla_p)=\int d^3q\phi(\mathbf{q})\exp(-\mathbf{q}\cdot\nabla_p).$$

By (2.68), the corresponding operator in the free-particle representation is

$$\phi^{\pm}=\phi\left(i\hbar\nabla_p-\frac{i\hbar}{2}\tau_1\frac{c^2\mathbf{p}}{E_p^2}\right)$$
$$=\int d^3q\phi(\mathbf{q})\exp\left[-\mathbf{q}\cdot\left(\nabla_p-\tfrac{1}{2}\tau_1\frac{c^2\mathbf{p}}{E_p^2}\right)\right].$$

We investigate only the lowest order real and virtual effects of the $\mathbf{x}^{(-)}$ operator. In this case we may write

$$\mathbf{x}^{(-)}=(-i\hbar/2)(\tau_1)(\mathbf{p}/m^2c^2),$$

so that the commutator,

$$[x_i^{(+)},x_k^{(-)}]=\tfrac{1}{2}\tau_1(\hbar/mc)^2\delta_{ik},$$

commutes with both $x^{(+)}$ and $x^{(-)}$. For such a pair, the following identity holds:

$$e^{a+b}=e^ae^be^{-\frac{1}{2}[a,b]}.$$

Applying this result to the exponential in the equation for $\phi^{\pm}$ we obtain

$$\phi^{\pm}=\int d^3q\phi(\mathbf{q})\exp(-\mathbf{q}\cdot\nabla_p)\exp\left[\frac{\tau_1}{4m^2c^2}(q^2+2\mathbf{p}\cdot\mathbf{q})\right].$$

The last term contains real and virtual effects of the Zitterbewegung. Expansion of the exponential gives

$$\phi^{\pm}=\int d^3q\phi(\mathbf{q})\left(1+\frac{(2\mathbf{p}\cdot\mathbf{q}-q^2)^2}{32(mc)^4}\right)\exp(-\mathbf{q}\cdot\nabla_p)$$
$$+\tau_1\int d^3q\phi(\mathbf{q})\frac{2\mathbf{p}\cdot\mathbf{q}-q^2}{4m^2c^2}\exp(-\mathbf{q}\cdot\nabla_p)+\cdots.$$

If therefore $\phi^{\pm}$ operates on the wave function $\Psi^{\pm}(\mathbf{p})$, we obtain, using (2.46) and dropping the τ_1 term,

$$\phi^{\pm}\Psi^{\pm}(\mathbf{p})=\int d^3q\phi(\mathbf{q})\left[1+\frac{(2\mathbf{p}\cdot\mathbf{q}-q^2)^2}{32m^4c^4}+\cdots\right]\Psi^{\pm}(\mathbf{p}-\mathbf{q})$$
$$=\int d^3q\phi(\mathbf{p}-\mathbf{q})\left[1+\frac{(p^2-q^2)^2}{32m^4c^4}\right]\Psi^{\pm}(\mathbf{q})+\cdots,$$

in complete agreement with previous results (2.56).

3. WAVE MECHANICS OF CHARGED SPIN ½ PARTICLES—THE DIRAC EQUATION[12]

This formalism is so well known that a word of explanation is in order. First we *derive* the Dirac equation as the unique generalization of the nonrelativistic Pauli equations. This seems to us a more satisfying procedure than the customary Dirac factorization. Secondly, we emphasize the parallels to the structure of the spin zero case discussed in the preceding section. However the derivation presented below is a digression both in spirit and content from the main purposes of this paper so that in a first reading Secs. A, B, and C may be omitted.

Let us first consider the derivation of the Dirac equation. We start from the nonrelativistic situation in which the wave function, φ, has two components corresponding to the two independent orientations of the spin and satisfies the Pauli equation,[13]

$$i\hbar\frac{\partial\varphi}{\partial t}=\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2+e\phi\varphi-\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\mathbf{H}\varphi, \tag{3.1}$$

where $\boldsymbol{\sigma}$ are the usual Pauli spin operators. This should be contrasted with the spin zero case where the nonrelativistic one particle equation requires only a single component wave function and where, as a consequence, the relativistic generalization is relatively easy (see (2.6)). Here we need to generalize the concept of spin and of spin space. We find that once this is done it is easier to continue the discussion in spin space and ask for covariant equations in that space. It turns out that there is only one (apart from those which are connected to this one by a canonical transformation) and that its nonrelativistic limit is (3.1).

A. Dirac Matrices[14]

We start with the nonrelativistic result that the Pauli-spin operators are proportional to the infinitesimal rotation operator $\mathbf{I}$ in spin space

$$\mathbf{I}=\tfrac{1}{2}\boldsymbol{\sigma}, \tag{3.2}$$

where $\boldsymbol{\sigma}$ are the usual Hermitian Pauli-spin operators. That is under a spatial rotation of angle $\boldsymbol{\omega}$ the nonrelativistic Schroedinger wave function ψ transforms to ψ' as follows:

$$\psi=\exp[-i(\mathbf{L}+\tfrac{1}{2}\boldsymbol{\sigma})\cdot\boldsymbol{\omega}]\psi', \tag{3.3}$$

where $\mathbf{L}$ is the orbital angular momentum in units of $\hbar$: $\mathbf{L}=1/\hbar(\mathbf{x}\times\mathbf{p})$.

From the point of view of the Lorentz transformation, the rotation operator $\mathbf{I}$ is a pseudovector constructed

[12] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928). The Dirac equation is discussed in many books and monographs, for example, by W. Pauli in the *Handbuch der Physik*, H. Geiger and K. Scheel, editors (Verlag Julius Springer, Berlin, 1933), second edition, Vol. 24, Part I.

[13] W. Pauli, Z. Physik **43**, 601 (1927).

[14] The algebra of the Dirac matrices is discussed by W. Pauli, Ann. inst. Henri Poincaré **6**, 109 (1936); R. H. Good, Jr., Revs. Modern Phys. **27**, 187 (1955).

from the spatial components of an antisymmetric tensor of second rank, $I_{\mu\nu}$. Turning to $\boldsymbol{\sigma}$ we see that there must be a tensor $\sum_{\mu\nu}$ for which $\boldsymbol{\sigma}$ forms just the spatial part. Therefore,

$$\sum_{\mu\nu}=-\sum_{\nu\mu}$$
$$\boldsymbol{\sigma}=(\sum_{23},\sum_{31},\sum_{12}).$$

To complete $\sum_{\mu\nu}$ we introduce the vector operator $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}=(\sum_{14},\sum_{24},\sum_{34}). \tag{3.4}$$

We note here that $\sum_{\mu\nu}F_{\mu\nu}$ ($F_{\mu\nu}$=electric and magnetic field intensities) forms an invariant, which is the correct generalization of the $\boldsymbol{\sigma}\cdot\mathbf{H}$ interaction in the Pauli equation.

The properties of $\boldsymbol{\alpha}$ can be completely determined from the Lorentz transformation connecting the properties of $\boldsymbol{\sigma}$ in one Lorentz frame with the properties of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ in another frame.

The defining properties of $\boldsymbol{\sigma}$ are

$$\sigma_i^2=1, \tag{3.5a}$$
$$\sigma_1\sigma_2=-\sigma_2\sigma_1=i\sigma_3 \tag{3.5b}$$
(and cyclic permutations).

Passing to a rotated coordinate system, the components of $\boldsymbol{\sigma}$ transform like a space (pseudo)-vector. If the properties (3.5) of $\boldsymbol{\sigma}$ are stated in the covariant form

$$\sigma_i\sigma_k+\sigma_k\sigma_i=2\delta_{ik},$$
$$(\boldsymbol{\sigma}\times\boldsymbol{\sigma})=2i\boldsymbol{\sigma}, \tag{3.6a,b}$$

the invariance of these properties under an orthogonal transformation becomes self-evident, so that (3.6) is also satisfied by the

$$\sigma_i'=\sum_k a_{ik}\sigma_k \tag{3.7}$$

(a_{ik} being the orthogonal matrix relating the two coordinate systems).[15]

Any vector V in spin space satisfies, as a consequence of (3.3), the commutation relations,[16]

$$[\sigma_i,V_i]=0 \quad [\sigma_i,V_j]=-[\sigma_j,V_i]=2iV_k \quad (i,j,k \text{ cyclic}),$$

an equation, which in terms of the $\sum_{ik}$, may be rewritten as

$$[V_i,\sum_{jk}]=-2i(\delta_{ij}V_k-\delta_{ik}V_j). \tag{3.8}$$

Incidentally, in nonrelativistic theory, the only vector in spin space is a multiple of $\boldsymbol{\sigma}$ itself. This is no longer so, on passing from the three-dimensional rotation group to the Lorentz group.

Under a Lorentz transformation from a reference system at rest to one (denoted by a prime) moving with a velocity v in the 1 direction we have

$$\sigma_1'=\sigma_1,$$
$$\sigma_2'=(1-v^2/c^2)^{-\frac{1}{2}}[\sigma_2+i(v/c)\alpha_3], \tag{3.9}$$
$$\sigma_3'=(1-v^2/c^2)^{-\frac{1}{2}}[\sigma_3-i(v/c)\alpha_2].$$

We now require that in the new system relations (3.5) still hold. Then from $\sigma_2'\sigma_3'=i\sigma_1'=i\sigma_1$, we have

$$i(1-v^2/c^2)\sigma_1=i\sigma_1+i(v/c)(\sigma_3\alpha_3-\alpha_2\sigma_2)+(v^2/c^2)\alpha_3\alpha_2.$$

Equating the coefficients of each power of v/c on both sides of this equation we obtain

$$\sigma_3\alpha_3=\alpha_2\sigma_2,$$

and

$$\sigma_1=i\alpha_3\alpha_2. \tag{3.10}$$

These relations hold for all cyclic permutations of the indices as will be true of the relations obtained immediately below. Similarly from $(\sigma_2')^2=1$ we find

$$\alpha_3^2=1, \tag{3.11}$$
$$\sigma_2\alpha_3+\alpha_3\sigma_2=0. \tag{3.12}$$

We now have a sufficient number of relations to determine the commutation rules of α_k with $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$. For example, from (3.10), and (3.12) we have

$$(\alpha_1\alpha_3)\alpha_3+\alpha_3(\alpha_1\alpha_3)=0.$$

It follows that

$$\alpha_1\alpha_3\alpha_3^2+\alpha_3\alpha_1\alpha_3^2=0,$$

and with (3.11) that

$$\alpha_1\alpha_3+\alpha_3\alpha_1=0. \tag{3.13}$$

Clearly (3.11) and (3.13) may be summed up in the general relation

$$\alpha_i\alpha_k+\alpha_k\alpha_i=2\delta_{ik} \tag{3.14}$$

for all $i,k=(1,2,3)$.

In a similar fashion (3.10) and its cyclic permutations are summed up [with the help of (3.14)] as

$$\alpha_i\alpha_j=-\alpha_j\alpha_i=i\sigma_k \quad (i,j,k \text{ cyclic}). \tag{3.15}$$

To evaluate the commutator of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$, we notice that $\boldsymbol{\alpha}$ is a 3-vector in spin space; consequently (3.8) gives

$$[\alpha_i,\sum_{jk}]=-2i(\delta_{ij}\alpha_k-\delta_{ik}\alpha_j). \tag{3.16}$$

We can verify this relation to be a consequence of (3.15). For instance

$$[\sigma_1,\alpha_2]=i\alpha_3\alpha_2\cdot\alpha_2+i\alpha_2\cdot\alpha_2\alpha_3=2i\alpha_3.$$

Having completely specified the properties of the rotation-operator $\sum_{\mu\nu}$ in spin space, we now prove the important statement that there exists a *unique* four vector γ_μ in spin space. Uniqueness here expresses the fact that the algebraic properties of γ_μ so pin down the operator that in an irreducible matrix representation of the operator in spin space any two vector γ_μ and γ_μ' must be numerical multiples of each other except for a

[15] Clearly the transformation (3.7) changes the *representation* of the $\boldsymbol{\sigma}$ matrices. The transformation $\exp(-\frac{1}{2}i\boldsymbol{\sigma}\cdot\boldsymbol{\omega})$ (3.3) just reestablishes the original matrix representation: $\sigma_i'=\sum a_{ik}(\boldsymbol{\omega})\sigma_k=\exp(1/2i\boldsymbol{\sigma}\cdot\boldsymbol{\omega})\sigma_i\exp(-1/2i\boldsymbol{\sigma}\cdot\boldsymbol{\omega})$.

[16] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1935), p. 59.

possible unitary transformation. The algebraic properties of γ_μ are shown to be

$$\gamma_\mu\gamma_\nu=-\gamma_\nu\gamma_\mu=i\sum_{\mu\nu} \quad (\mu\neq\nu) \tag{3.17a}$$

and, for $\mu=\nu$: all γ_μ^2 are equal and a multiple of the unit matrix, so that we may define a normalized γ_μ as obeying

$$\gamma_\mu^2=I \quad (\mu=1, 2, 3, 4), \tag{3.17b}$$

where I is the unit matrix.

To prove these remarks we perform a general infinitesimal Lorentz transformation on γ_μ and from this obtain the commutation relations between $\sum_{\alpha\beta}$ and γ_μ. In other words we evaluate[15]

$$\gamma_{\mu'}=\exp\left(\frac{i}{2}\sum_{\alpha\beta}\omega_{\alpha\beta}\right)\gamma_\mu\exp\left(-\frac{i}{2}\sum_{\alpha\beta}\omega_{\alpha\beta}\right)$$
$$=\sum a_{\mu\nu}(\omega_{\alpha\beta})\gamma_\nu, \tag{3.18}$$

where $a_{\mu\nu}$ form the Lorentz transformation. Rather than work this out in detail we remark that the result will be the covariant generalization of (3.8) which is

$$[\gamma_\mu,\sum_{\alpha\beta}]=-2i[\delta_{\mu\alpha}\gamma_\beta-\delta_{\mu\beta}\gamma_\alpha]. \tag{3.19}$$

We now examine the consequences of (3.19) which follow from the properties of $\sum_{\alpha\beta}$. For example, since $(\sum_{\mu\nu})^2$ (not summed over μ or ν) is the unit matrix, we have (no summation over μ)

$$[(\gamma_\mu,(\textstyle\sum_{\mu\nu})^2]=0=\sum_{\mu\nu}[\gamma_\mu,\sum_{\mu\nu}]+[\gamma_\mu,\sum_{\mu\nu}]\sum_{\mu\nu}.$$

Substituting from (3.19) we find

$$\{\gamma_\mu,\sum_{\mu\nu}\}=0, \tag{3.20}$$

where

$$\{A,B\}=AB+BA.$$

We can also prove that

$$[\gamma_\mu^2,\sum_{\alpha\beta}]=0. \tag{3.21}$$

For $\mu=\alpha$, and $\mu=\beta$, this follows from (3.19). For $\mu=\alpha$, then

$$[\gamma_\alpha^2,\sum_{\alpha\beta}]=\gamma_\alpha\{\gamma_\alpha,\sum_{\alpha\beta}\}-\{\gamma_\alpha,\sum_{\alpha\beta}\}\gamma_\alpha$$

which vanishes because of (3.20) proving (3.21). We may, however, also write the above commutator in terms of commutators rather than anticommutators. Then

$$[\gamma_\alpha^2,\sum_{\alpha\beta}]=\gamma_\alpha[\gamma_\alpha,\sum_{\alpha\beta}]+[\gamma_\alpha,\sum_{\alpha\beta}]\gamma_\alpha=0.$$

Substituting (3.19) we obtain

$$\{\gamma_\alpha,\gamma_\beta\}=0 \quad (\alpha\neq\beta) \tag{3.22}$$

proving the first part of (3.17a).

The main results so far are (3.21) and (3.22). The first of these shows that γ_μ^2 commutes with all rotations and is, therefore, a scalar, which we call I,

$$\gamma_\mu^2=I \quad \text{for all } \mu. \tag{3.23}$$

With this result, we may write (3.22) in the covariant form,

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2I\delta_{\mu\nu}. \tag{3.24}$$

Although (3.23) is quite sufficiently established, we shall add a simple algebraic proof of the identity of all operators γ_μ^2. From (3.19) and (3.20) we derive

$$\gamma_\alpha\sum_{\alpha\beta}=-i\gamma_\beta,$$
$$\gamma_\beta\sum_{\alpha\beta}=i\gamma_\alpha,$$

and multiplying the first from the left with γ_β, the second with γ_α, we get with (3.22)

$$\gamma_\beta^2=\gamma_\alpha^2,$$

which proves (3.23).

We now show that I, in addition to commuting with all the $\sum_{\mu\nu}$, also commutes with all the γ_μ. This is, in fact, trivial, since

$$[I,\gamma_\mu]=[\gamma_\mu^2,\gamma_\mu]=0.$$

On the basis of this result, we can get an irreducible representation of the γ_μ and $\sum_{\mu\nu}$ matrices by choosing I to be the *unit* matrix.

The $\sum_{\mu\nu}$ can then be expressed in terms of the γ_μ: From (3.19) we have (no summation over α)

$$\gamma_\alpha\sum_{\alpha\beta}\gamma_\alpha-\sum_{\alpha\beta}\gamma_\alpha^2=-2i\gamma_\beta\gamma_\alpha$$

or with (3.20), (3.22), and (3.23)

$$\sum_{\alpha\beta}I=-i\gamma_\alpha\gamma_\beta.$$

Since I equals unit matrix, it follows that

$$\sum_{\alpha\beta}=-i\gamma_\alpha\gamma_\beta=1/2i[\gamma_\alpha,\gamma_\beta].$$

Thus we have derived all the familiar results about the Dirac matrices, and we shall simply take over the well-known conclusion that the γ_μ may be represented in terms of 4×4 matrices.

B. The Dirac Equation

The existence of a four vector in spin space provides us with the possibility of constructing a first-order covariant wave equation for a spin $\frac{1}{2}$ particle,

$$(\textstyle\sum_\mu D_\mu\gamma_\mu+mc/\hbar)\psi=0. \tag{3.25}$$

It is not immediately evident that this equation is the "unique relativistic generalization" of the Pauli nonrelativistic wave equation (3.1); but that this is so becomes apparent shortly. To obtain this equation in Hamiltonian form, we multiply through with $\hbar c\gamma_4$ and use $i\gamma_4\gamma_k=\alpha_k$, a relation which follows from (3.3) and (3.17a). This leads to

$$[c\boldsymbol{\alpha}\cdot(\mathbf{p}-e/c\mathbf{A})+\gamma_4mc^2+e\phi]\psi=i\hbar\partial\psi/\partial t. \tag{3.26}$$

The operator on the left is the Dirac Hamiltonian. This reordering of the equation satisfied by ψ is analogous to the one performed for the Klein-Gordon equation in Sec. 2B.

We do not derive the familiar representations of γ_μ because these derivations are available in many places.[12,14] The results are that ψ is a unicolumnar matrix of four elements, while the γ_μ, $\boldsymbol{\alpha}$, $\sum_{\mu\nu}$ matrices are 4×4 matrices. To obtain an analogy with the spin zero equation we use the "split" notation

$$\psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}, \tag{3.27}$$

where φ and χ are two component amplitudes which play a role exactly equivalent to the φ and χ of the spin zero theory, on using a representation in which γ_4 is diagonal, (γ_4 becomes the analog of the τ_3 of the spin 0 case)

$$\gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \alpha_k=\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix}. \tag{3.28}$$

With this, (3.26) may be written as a set of two coupled two-component equations,

$$\begin{aligned} i\hbar(\partial\varphi/\partial t)&=c\boldsymbol{\sigma}\cdot(\mathbf{p}-e/c\mathbf{A})\chi+(mc^2+e\phi)\varphi,\\ i\hbar(\partial\chi/\partial t)&=c\boldsymbol{\sigma}\cdot(\mathbf{p}-e/c\mathbf{A})\varphi+(-mc^2+e\phi)\chi. \end{aligned} \tag{3.29}$$

The relation of (3.29) to the Pauli equation can now be established. Adding the rest energy term to the *nonrelativistic* Pauli equation (3.1), the latter reads

$$i\hbar\frac{\partial\varphi}{\partial t}=\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2\varphi+mc^2\varphi+e\phi\varphi+\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\mathbf{H}\varphi. \tag{3.30}$$

But using the well-known relations,

$$\left[\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\right]\left[\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\right]=\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2-\frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\mathbf{H},$$

we can write (3.30) exactly as a set of two coupled equations:

$$\begin{aligned} i\hbar\frac{\partial\varphi}{\partial t}&=c\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\chi+mc^2\varphi+e\phi\varphi,\\ mc^2\chi&=c\boldsymbol{\sigma}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\varphi-mc^2\chi. \end{aligned} \tag{3.31}$$

The similarity between (3.29) and (3.31) is evident. The main difference, apart from the omission of the small electrostatic term, lies in the suppression of a degree of freedom which results from specifying the time dependence of χ to be $\exp[-i(mc^2/\hbar)t]$, thus reducing the second equation in (3.29) to a subsidiary condition.

C. Physical Interpretation

We do not derive the familiar expressions which give the physical content of the Dirac equation, but rather quote those that we need in the following discussion, in which the common elements of the scalar and spin $\frac{1}{2}$ equations are emphasized. In both, the effect of the relativistic generalizations is to *double* the number of components of the wave function. In the scalar case the nonrelativistic equation has one component, the relativistic two; in the spin $\frac{1}{2}$ case the nonrelativistic has two, the relativistic, four components. This doubling of degrees of freedom in both cases gives rise to formally negative energy solutions; in both cases the negative energy solutions are related to the positive energy solutions of a charge-conjugate problem: To get the charge conjugate equation, for the Dirac case, we take the complex conjugate of (3.29) and observe that

$$\mathbf{p}^*=(\hbar/i\nabla)^*=-\mathbf{p}$$

and $\boldsymbol{\sigma}^*=\boldsymbol{\sigma}^T$, the transposed $\boldsymbol{\sigma}$, if we assume the usual representation of the σ as Hermitian matrices. Then

$$i\hbar(\partial\chi^*/\partial t)=c\boldsymbol{\sigma}^T\cdot\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\varphi^*+mc^2\chi^*-e\phi\chi^*,$$

$$i\hbar(\partial\varphi^*/\partial t)=c\boldsymbol{\sigma}^T\cdot\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\chi^*-mc^2\varphi^*-e\phi\varphi^*.$$

Since the transposed Pauli matrices σ^T satisfy

$$\sigma_x{}^T=\sigma_x;\quad \sigma_y{}^T=-\sigma_y;\quad \sigma_z{}^T=\sigma_z$$

we have, from (3.5b):

$$\sigma_y\sigma_i{}^T\sigma_y=-\sigma_i.$$

The wave functions

$$\begin{aligned} \varphi_c&=-i\sigma_y\chi^*,\\ \chi_c&=i\sigma_y\varphi^*, \end{aligned}$$

satisfy

$$\begin{aligned} i\hbar\frac{\partial\varphi_c}{\partial t}&=c\boldsymbol{\sigma}\cdot\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\chi_c+(mc^2-e\phi)\varphi_c,\\ i\hbar\frac{\partial\chi_c}{\partial t}&=c\boldsymbol{\sigma}\cdot\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\varphi_c-(mc^2+e\phi)\chi_c, \end{aligned} \tag{3.32}$$

which differs from (3.29) by a change in sign of the electric charge.

A discussion exactly analogous to that in Sec. 2 shows that (3.31) associates a positive energy solution of (3.32) with a negative energy solution of (3.29) and that, if the positive energy solution has a total charge of $+1$, the charge conjugate solution, as well as the corresponding negative energy solution, has the total charge -1. This again establishes the physical meaning of the negative energy solutions. For a free electron, if the momentum for

$$\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

is $\mathbf{p}$, then the momentum for the charge conjugate function is $-\mathbf{p}$. The effect on the spin operator is also

to turn the spin around so that the relation between spin and momentum for the charge conjugate solution is the same as for the original wave function.

D. Free-Particle Representation[4]

Again, as in the spin 0 case, it is useful, for a discussion of the nonrelativistic limit, and of the Zitterbewegung, to go to a representation, in which the free-particle Hamiltonian

$$H_0(\mathbf{p})=c(\boldsymbol{\alpha}\cdot\mathbf{p})+mc^2\beta \quad \beta=\gamma_4 \tag{3.33}$$

is diagonal. Since $H_0^2(p)=E_p^2$, the columns of the 4×4 matrix

$$H_0(\mathbf{p})\beta+E_p \,^{17} \tag{3.34}$$

are solutions of the free-particle Dirac equation

$$[H_0(\mathbf{p})-E]\Psi(\mathbf{p})=0$$

with eigenvalues E_p, E_p, $-E_p$, $-E_p$. To normalize these solutions, we divide (3.34) by $[2E_p(E_p+mc^2)]^{\frac{1}{2}}$.

With this normalization, (3.34) furnishes the matrix which describes the transformation to the free-particle representation,

$$\begin{aligned} U(\mathbf{p})&=\frac{H_0(\mathbf{p})\beta+E_p}{[2E_p(E_p+mc^2)]^{\frac{1}{2}}},\\ U^\dagger(\mathbf{p})&=\frac{\beta H_0(\mathbf{p})+E_p}{[2E_p(E_p+mc^2)]^{\frac{1}{2}}}. \end{aligned} \tag{3.35}$$

Again exactly as in the spin zero case, we make use of $U(\mathbf{p})$ to transform the general momentum space Dirac equation

$$i\hbar\partial\psi(\mathbf{p},t)/\partial t=\left\{c\boldsymbol{\alpha}\cdot\left[\mathbf{p}-\frac{e}{c}\mathbf{A}(i\hbar\nabla_p)\right]+mc^2\beta+e\phi(i\hbar\nabla_p)\right\}\psi(\mathbf{p},t),$$

into the free-particle representation,

$$\psi(\mathbf{p})=U(\mathbf{p})\psi^{\pm}(\mathbf{p}),$$

where

$$\psi^{\pm}(\mathbf{p})=\begin{pmatrix}u(\mathbf{p})\\ v(\mathbf{p})\end{pmatrix}.$$

Then

$$\begin{aligned} i\hbar\partial\psi^{\pm}/\partial t&=U^{-1}(\mathbf{p})H(\mathbf{p},i\hbar\nabla_p)U(\mathbf{p})\psi^{\pm}(\mathbf{p})\\ &=H^{\pm}(\mathbf{p},i\hbar\nabla_p)\psi^{\pm}(\mathbf{p}). \end{aligned} \tag{3.36}$$

Before writing (3.36) in more detail, we recall a procedure applied in connection with the discussion of the position operator $\mathbf{x}=i\hbar\nabla_p$ in Sec. 3. We noted that $\mathbf{x}$ is transformed into $\mathbf{x}^{\pm}=U^{-1}\mathbf{x}U$ by this change of representation, and we showed that for any function of $\mathbf{x}$ we may write

$$U^{-1}(\mathbf{p})f(i\hbar\nabla_p)U(\mathbf{p})=f(U^{-1}i\hbar\nabla_pU)=f(\mathbf{x}^{\pm}).$$

[17] A similar result holds for the free-particle solutions $\Psi^{(\pm)}$ in the scalar boson case.

Using this technique to write out $H^{\pm}$ in (3.36), we have

$$H^{\pm}=\beta E_p-e(U^{-1}\boldsymbol{\alpha}U)\cdot\mathbf{A}(\mathbf{x}^{\pm})+e\phi(\mathbf{x}^{\pm}). \tag{3.36'}$$

With the help of (3.35) we find that

$$\begin{aligned} \boldsymbol{\alpha}^{\pm}&=U^{-1}(\mathbf{p})\boldsymbol{\alpha}U(\mathbf{p})\\ &=\boldsymbol{\alpha}+\beta\frac{c\mathbf{p}}{E_p}-\boldsymbol{\alpha}\cdot\mathbf{p}\frac{c^2\mathbf{p}}{E_p(E_p+mc^2)}, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} \mathbf{x}^{\pm}&=U^{-1}(\mathbf{p})i\hbar\nabla_pU(\mathbf{p})\\ &=i\hbar\nabla_p+i\hbar\left[\frac{ic(\boldsymbol{\sigma}\times\mathbf{p})}{2E_p(E_p+mc^2)}+\frac{\boldsymbol{\alpha}\beta c}{2E_p}-\frac{(\boldsymbol{\alpha}\cdot\mathbf{p})\beta\mathbf{p}c^3}{2E_p^2(E_p+mc^2)}\right]. \end{aligned} \tag{3.38}$$

By the same argument as given for the spin zero case, (3.35) provides a splitting of $\mathbf{x}$ into an even and odd part: To the approximation needed below, we have

$$\mathbf{x}^{\pm}=(\mathbf{x}^{\pm})^{(+)}+(\mathbf{x}^{\pm})^{(-)},$$

$$(\mathbf{x}^{\pm})^{(+)}\simeq i\hbar\left(\nabla_p+i\frac{\boldsymbol{\sigma}\times\mathbf{p}}{(2mc)^2}\right);\quad (\mathbf{x}^{\pm})^{(-)}\simeq i\hbar\frac{\boldsymbol{\alpha}\beta}{2mc}. \tag{3.39}$$

Thus $(\mathbf{x}^{\pm})^{(+)}$ consists of the displacement operator in $\mathbf{p}$ space and of a spin flip part.

However, from the point of view of discussing the relativistic corrections to the Pauli equation, and particularly the Zitterbewegung, it is more convenient not to split $\mathbf{x}^{\pm}$ into its even and odd parts but rather to write

$$\mathbf{x}^{\pm}=\mathbf{x}'+\delta\mathbf{x},$$

where

$$\mathbf{x}'=i\hbar\nabla_p;\quad \delta\mathbf{x}=i\hbar\left[\frac{i(\boldsymbol{\sigma}\times\mathbf{p})}{(2mc)^2}+\frac{\boldsymbol{\alpha}\beta}{2mc}\right], \tag{3.40}$$

and to employ the representation in which $\mathbf{x}'$ rather than $\mathbf{x}$ is diagonal.

With this notation, the equations which follow hold in either the momentum representation

$$\mathbf{p}=\mathbf{p}' \quad \mathbf{x}'=i\hbar\nabla_p \tag{3.41}$$

or in an $\mathbf{x}$ representation, in which $\mathbf{x}'$ is diagonal

$$\mathbf{p}=i\hbar\nabla_p;\quad \mathbf{x}'=\mathbf{x}. \tag{3.41b}$$

We are now in a position to write down the transformed Hamiltonian $H^{\pm}$. As our main interest is the nonrelativistic limit and the first relativistic corrections to it, we retain only the necessary terms:

$$H^{\pm}\simeq\beta E_p-e\left(\boldsymbol{\alpha}+\beta\frac{\mathbf{p}c}{E_p}\right)\cdot\mathbf{A}(\mathbf{x}+\delta\mathbf{x})+e\phi(\mathbf{x}+\delta\mathbf{x}). \tag{3.42}$$

The free-particle Dirac equation is rigorously decoupled in this representative, and reads

$$i\hbar\partial u/\partial t = E_p u,$$
$$i\hbar\partial v/\partial t = -E_p v.$$

But again as in the spin zero case, there exists a weak field situation, in which the two amplitudes u and v of $\psi^{\pm}$ in (3.36) are only very weakly coupled. This is especially true for the nonrelativistic limit, where we may assume that only either u or v is nonzero.

E. Nonrelativistic Limit and Zitterbewegung

For free particles we can easily construct the velocity operators associated with both parts of the position operator,

$$\begin{aligned}(\mathbf{v}^{\pm})^{(+)} &= i/\hbar[\beta E_p, (\mathbf{x}^{\pm})^{(+)}] = \beta\mathbf{p}c^2/E_p,\\ (\mathbf{v}^{\pm})^{(-)} &= i/\hbar[\beta E_p, (\mathbf{x}^{\pm})^{(-)}]\\ &= c\left[\boldsymbol{\alpha} - \frac{(\boldsymbol{\alpha}\cdot\mathbf{p})c^2\mathbf{p}}{E_p(E_p+mc^2)}\right].\end{aligned} \tag{3.43}$$

Clearly $(\mathbf{v}^{\pm})^{(+)} + (\mathbf{v}^{\pm})^{(-)} = c\boldsymbol{\alpha}^{\pm}$, the transformed velocity operator, since in the original representation $\mathbf{v} = c\boldsymbol{\alpha}$.

$\mathbf{v}^{(-)}$ is the velocity operator of the Zitterbewegung. For purely positive (or purely negative) states in the field free case, $\mathbf{v}^{(-)}$ has, of course, zero expectation value. As in the spin zero case, here again it is impossible to construct purely positive packets that are localized to any better than $\hbar/mc$. The operator $\mathbf{x}^{(+)}$ in x space, is a nonlocal operator, given by an integral kernel [compare (2.63)].

$$\mathbf{x}^{(+)} = \mathbf{x}\delta(\mathbf{x}-\mathbf{x}') - \mathbf{X}^{(-)}(\mathbf{x}-\mathbf{x}'),$$
$$\mathbf{X}^{(-)}(\mathbf{x}-\mathbf{x}') = \frac{i\hbar}{(2\pi\hbar)^3}\int d^3p e^{i\mathbf{p}/\hbar(\mathbf{x}-\mathbf{x}')} \times\left[\frac{\boldsymbol{\alpha}\beta c}{2E_p} - \frac{(\boldsymbol{\alpha}\cdot\mathbf{p})\beta\mathbf{p}c^3}{2E_p^2(E_p+mc^2)}\right].$$

As in the spin zero case we can construct eigenfunctions of $\mathbf{x}^{(\pm)}$ and find that they are packets with a width $\hbar/mc$.

As soon as an external field is turned on, the Zitterbewegung comes into play. Let us discuss this effect again for the case of a magnetic field and for the case of a central electrostatic field.

(a) *Magnetic Field*

Assuming for simplicity a constant field with $\mathbf{A}(\mathbf{x}) = \frac{1}{2}(\mathbf{H}\times\mathbf{x})$, the magnetic coupling term in (3.41) is

$$-\frac{e}{2}\mathbf{H}\cdot[(\mathbf{x}'+\delta\mathbf{x})\times((\mathbf{v}^{\pm})^{(+)} + (\mathbf{v}^{\pm})^{(-)}].$$

It is easily seen that

$$(\mathbf{v}^{\pm})^{(+)} = (d\mathbf{x}'/dt), \quad \mathbf{v}^{(-)} = (d/dt)(\delta\mathbf{x}).$$

The magnetic energy consists of four separate terms. The leading term is

$$-(e/2)\mathbf{H}\cdot\left(\mathbf{x}'\times\frac{d\mathbf{x}'}{dt}\right) = -(e/2mc)\frac{mc^2}{E_p}\mathbf{H}\cdot\mathbf{L}, \tag{3.44}$$

$\mathbf{L}$ being the orbital angular momentum $\mathbf{L} = (i\hbar\nabla_p\times\mathbf{p})$.

As in the spin zero theory, this orbital magnetic coupling is reduced by a factor (mc^2/E_p) as compared to the nonrelativistic limit (2.58).

Going to the next terms, the "cross terms" in $(\mathbf{x}'+\delta\mathbf{x})\times\{[(d\mathbf{x}'/dt)+(d/dt)(\delta\mathbf{x})]\}$ have a vanishing nonrelativistic limit, and need not be discussed here. The last term, however, becomes

$$-\frac{e}{2}\mathbf{H}\cdot\left(\delta\mathbf{x}\times\frac{d}{dt}\delta\mathbf{x}\right) \simeq -\frac{e}{2}\mathbf{H}\cdot\left(\frac{i\hbar\boldsymbol{\alpha}\beta}{2mc}\times\boldsymbol{\alpha}\right) \simeq -\frac{e}{2mc}\beta(\mathbf{H}\cdot\boldsymbol{\sigma}), \tag{3.45}$$

using $(\boldsymbol{\alpha}\times\boldsymbol{\alpha}) = 2i\boldsymbol{\sigma}$. This derivation shows clearly the connection of the spin magnetic moment with the odd parts of the position- and velocity-operators, and justifies to a certain extent the phrase that "the spin magnetic moment is the orbital moment of the Zitterbewegung."[18]

(b) *Electrostatic Field*

In this case we expand the electrostatic potential term in (3.41)

$$e\phi(\mathbf{x}) = e\phi(\mathbf{x}'+\delta\mathbf{x}).$$

In the approximation (3.40), $\mathbf{x}_i'$ and $\delta\mathbf{x}_i$ *commute*. This makes the situation simpler than in the spin zero case, as we may now write

$$\phi(\mathbf{x}) = \phi(\mathbf{x}') + \sum_i \delta\mathbf{x}_i\frac{\partial\phi}{\partial x_i} + \tfrac{1}{2}\sum\delta x_i\delta x_j\frac{\partial^2\phi}{\partial x_i\partial x_j} + \cdots.$$

Inserting (3.40), we get the following *even* terms (the odd ones vanish in the nonrelativistic limit):

$$\text{even part of } \phi(\mathbf{x}) \simeq \phi(\mathbf{x}') - \frac{\hbar}{(2mc)^2}(\boldsymbol{\sigma}\times\mathbf{p})\cdot\nabla\phi(\mathbf{x}') - \frac{\hbar^2}{8m^2c^2}\nabla^2\phi(\mathbf{x}').$$

This expansion produces in a straightforward manner the well-known correction terms, which for a central potential may be written as

$$\frac{e\hbar}{(2mc)^2}(\boldsymbol{\sigma}\cdot\mathbf{L})\frac{1}{r}\frac{d\phi}{dr} + \frac{2\pi\hbar^2}{(2mc)^2}\rho(r), \tag{3.46}$$

[18] K. Huang, Am. J. Phys. **20**, 479 (1952).

and are identified as the spin-orbit coupling and the Darwin term, respectively and where ρ is the charge density giving rise to ϕ. We have, in this last equation, gone back to an x-space representation employing relations (3.41b). The diagonal position operator thus introduced is called the "mean" position: the nonrelativistic limit to the Dirac equation is most conveniently expressed in terms of this mean position. To sum up, we have, to order $(v/c)^2$

$$i\hbar\frac{\partial\psi^{\pm}}{\partial t}=\left\{\beta\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^4}\right)+e\phi(\mathbf{x})-\frac{e}{2mc}\beta(\boldsymbol{\sigma}\cdot\mathbf{H})\right.$$
$$\left.+\frac{e\hbar}{(2mc)^2}(\boldsymbol{\sigma}\cdot\mathbf{L})\frac{1}{r}\frac{d\phi}{dr}+\frac{2\pi\hbar^2}{(2mc)^2}\rho(r)\right\}\psi^{\pm}. \quad (3.47)$$

4. CONNECTION WITH FIELD THEORY; NEUTRAL PARTICLES[19]

In the interpretation given in the preceding section, we have considered the Klein-Gordon and Dirac equations to be single particle equations in which the particle also possesses the charge degree of freedom. There are limitations to such an interpretation which arise when many particle phenomena intervene, as in the presence of strong fields. Associated with this real process there are also virtual processes responsible, for example, for vacuum polarization. It might be thought that the single particle interpretation might fail here. To obtain a deeper insight into these problems and other questions of interpretation of the single particle formalism, we turn to the more general and more powerful formalism of field quantization.

The complete equivalence of the wave-mechanical description in configuration space and the quantum field theoretical description in the nonrelativistic case is well known. We demonstrate here the equally well-known relations[20] in the *external field approximation* of relativistic single particle wave mechanics and relativistic field quantization. Whereas the analysis of this relation definitely settles certain questions as, for instance, the interpretation of the negative frequency solutions of the one particle problem as well as those mentioned in the preceding paragraph, we get little help in others like the question of the meaning of the "position operator" and the like. Concepts like the latter are foreign to the field theoretical formalism and become meaningful only in the limit of weak-external fields where a sufficiently close analogy to the nonrelativistic situation prevails.

This short digression into field theory will be most useful in connection with the problem of the description of intrinsically neutral particles (e.g., the π^0 meson). We do not obtain much guidance for this case from the discussion of Sec. 2.

Our program is therefore first to establish anew the relation between the single particle formalism and quantum field theory for charged particles and then to build up a neutral particle formalism by starting from the field theory version. This will enable us to write a one particle wave equation for the scattering of π mesons in an external field (e.g., the field of a nucleon or a nucleus) which includes the processes of charge exchange.

A. Field Theory of Charged Particles in an External Field

We discuss here only the problem of a particle in an external field.

The field Hamiltonian[21] $\mathbf{H}$ is obtained from the wave-mechanical expression for the energy,

$$\langle E\rangle=\int d^3x\Psi^*\tau_3\left[\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2(\tau_3+i\tau_2)+mc^2\tau_3+e\phi\right]\Psi$$

(for scalar bosons) (4.1a)

or

$$\langle E\rangle=\int d^3x\Psi^*\left[c\boldsymbol{\alpha}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2+mc^2\beta+e\phi\right]\Psi$$

(for fermions) (4.1b)[22]

by substituting the operator $\boldsymbol{\Psi}$ for the wave function Ψ

$$\boldsymbol{\Psi}=\sum_p\left[\Psi^{(+)}(\mathbf{p})a_p+\Psi^{(-)}(\mathbf{p})b_{-p}^*\right]\exp(i/\hbar\mathbf{p}\cdot\mathbf{x}). \quad (4.2)$$

The $\Psi^{(\pm)}$ are normalized momentum eigenstates of the wave-mechanical free-particle Hamiltonian,

$$\text{(bosons)}\quad\left[\frac{p^2}{2m}(\tau_3+i\tau_2)+mc^2\tau_3\right]\Psi^{(\pm)}(\mathbf{p})=\pm E_p\Psi^{(\pm)}(\mathbf{p}),$$

$$\text{(fermions)}\qquad[c(\boldsymbol{\alpha}\cdot\mathbf{p})+mc^2\beta]\Psi^{(\pm)}(\mathbf{p})=\pm E_p\Psi^{(\pm)}(\mathbf{p}).$$

Thus for bosons $\Psi^{(\pm)}$ are two component amplitudes; for fermions, four component. For the latter, a spin-index s ($s=+\frac{1}{2},-\frac{1}{2}$) must be added for a complete description, but we shall not write it out explicitly.

The operators a, a^*, b, b^* satisfy the commutation relations

$$\text{(bosons)}\quad[a_p,a_{p'}^*]=[b_p,b_{p'}^*]=\delta_{pp'} \quad (4.3)$$

all other commutators zero

$$\text{(fermions)}\quad\{a_p,a_{p'}^*\}=\{b_p,b_{p'}^*\}=\delta_{pp'} \quad (4.4)$$

all other anticommutators zero. In this way, $(a_p^*a_p)$

[19] Some general references to quantum field theory: W. Pauli, Revs. Modern Phys. **13**, 203 (1941); J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Press, Cambridge, 1955); Schweber, Bethe, and deHoffman, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1955), Vol. 1.

[20] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, New York, 1947), third edition, Chap. X; V. Fock, Physik Z. Sowjetunion **6**, 1425 (1934); J. Pirenne, Physica **15**, 1023 (1949).

[21] To avoid confusion all field operators are in boldface.

[22] These expressions hold independently of the single particle interpretation; $\langle E\rangle$ is identical with $-\int T_{44}d^3x$, where $T_{\mu\nu}$ is the energy-momentum density associated with the wave equation in question.

and $(b_p^* b_p)$ are the operators of "occupation numbers" for positive and negative charge, respectively.

As a consequence of (4.3) and (4.4) the field equations for Ψ hold as operator equations with $i/\hbar[\mathbf{H},\Psi]$ equalling $\partial\Psi/\partial t$. Recall also that the choice of the commutator in (4.3) and anticommutator in (4.4) corresponds to limiting the eigenstates of the Hamiltonian H to those which are symmetrical (for boson) and antisymmetrical (for fermion) under exchange of two particles.

The eigenstates of $\mathbf{H}$ represent many particle situations. But since, in the external field approximation, these particles are not interacting, the problem of finding the eigenstates of $\mathbf{H}$ is essentially a one-particle problem. We now show that the formulation of this one-particle problem leads exactly to the wave-mechanical equations discussed in Secs. 2 and 3.

Let us now write down the Hamiltonian for an external electrostatic field ϕ. Inserting the operator (4.2) into the energy expression (4.1), we obtain the field Hamiltonian (apart from a constant)

$$\begin{aligned}\mathbf{H}=&\sum_p E_p(a_p^* a_p+b_p^* b_p)\\ &+e\sum_{pq}\phi(\mathbf{p}-\mathbf{q})\{(\mathbf{p}+|\mathbf{q}+)a_p^* a_q\\ &+(\mathbf{p}-|\mathbf{q}-)b_{-q}^* b_{-p}+(p+|q-)a_p^* b_{-q}^*\\ &+(p-|q+)a_q b_{-p}\},\end{aligned}\tag{4.5}$$

when we have used an abbreviation for certain coefficients: For bosons

$$\begin{aligned}(\mathbf{p}\pm|\mathbf{q}\pm)&=\Psi^{(\pm)}(\mathbf{p})^*\tau_3\Psi^{(\pm)}(\mathbf{q}),\\ (\mathbf{p}\pm|\mathbf{q}\mp)&=\Psi^{(\pm)}(\mathbf{p})^*\tau_3\Psi^{(\mp)}(\mathbf{q}),\end{aligned}$$

and for fermions

$$\begin{aligned}(\mathbf{p}\pm|\mathbf{q}\pm)&=\pm\Psi^{(\pm)}(\mathbf{p})^*\Psi^{(\pm)}(\mathbf{q}),\\ (\mathbf{p}\pm|\mathbf{q}\mp)&=\pm\Psi^{(\pm)}(\mathbf{p})^*\Psi^{(\mp)}(\mathbf{p}).\end{aligned}$$

With these definitions, the Hamiltonian (4.5) is the *same* for both bosons and fermions.

We now discuss the problem of constructing the eigenstates of $\mathbf{H}$. We consider first the vacuum state, Φ_0. There is a "naive" vacuum state, Φ_0^0, characterized by the absence of free particles

$$a_p\Phi_0^0=0;\quad b_p\Phi_0^0=0.$$

This state is not an eigenstate of $\mathbf{H}$, as one may verify by direct substitution. Correctly, the vacuum state Φ_0 is defined as the lowest eigenstate of $\mathbf{H}$, for zero value of momentum and charge. The charge is given by the operator

$$Q_F=e/2\int d^3x(\Psi^*\Psi-\Psi\Psi^*),\qquad\text{(for fermions)}$$

$$Q_B=e/2\int d^3x(\Psi^*\tau_3\Psi+\Psi(\Psi^*\tau_3)),\qquad\text{(for bosons)}$$

or, with (4.2)

$$Q_{F,B}=e\sum(a_p^* a_p-b_p^* b_p).\tag{4.6}$$

The commutation relations

$$\begin{aligned}Qa_p&=a_p(Q-e)\qquad Qb_p=b_p(Q+e)\\ Qa_p^*&=a_p^*(Q+e)\quad Qb_p^*=b_p^*(Q-e)\end{aligned}$$

show that a_p^* and b_p create a charge e, a_p and b_p^* destroy a charge e. Note that $a_p\Phi_0$ is not zero; i.e., in the presence of an interaction the zero charge, zero momentum state is not one for which the occupation numbers are all zero. Charges are present but are paired off so that the net charge is zero. This phenomenon is connected with the possibility of "polarization of the vacuum."

One "particle" states may now be described as states carrying *one* unit of charge, positive or negative. We call these states

$$\Phi_{1,n}\quad\text{and}\quad\Phi_{-1,n},$$

respectively. They are eigenstates of $\mathbf{H}$ with energy $\mathcal{E}_{1,n}$ and $\mathcal{E}_{-1,n}$, eigenstates of Q with eigenvalues $+e$ and $-e$, respectively, the subscript n denoting the various one-particle states. In the discussion below we often do not write it in explicitly. We also specify that we are not concerned with three (or more) real particles having a net charge of one unit. This is done by requiring that at a large distance from the interaction with the external field, where the particle is essentially free, the energy and momentum of the particle are related as usual, i.e., $E=(m^2c^4+p^2)^{\frac{1}{2}}$.

The question of the eigenvalues $\mathcal{E}_n$ leads back trivially to the eigenvalue problem discussed in Secs. 2 and 3: Exploiting the fact that both $\Phi_{1,n}$ and Φ_0 (the vacuum) are eigenstates of $\mathbf{H}$, with energies $\mathcal{E}_{1n}$ and 0 respectively, we have the relations,

$$(\Phi_0,[a_p,\mathbf{H}]\Phi_{1,n})=\mathcal{E}_{1n}(\Phi_0,a_p\Phi_{1,n}),\tag{4.7a}$$

$$(\Phi_0,[b_{-p}^*,\mathbf{H}]\Phi_{1,n})=\mathcal{E}_{1n}(\Phi_0,b_{-p}^*\Phi_{1,n}).\tag{4.7b}$$

The commutators on the left side are linear in a and b^*:

$$\begin{aligned}[a_p,\mathbf{H}]=&E_p a_p+e\sum_q\phi(\mathbf{p}+\mathbf{q})[a_q(\mathbf{p}+|\mathbf{q}+)\\ &+b_{-q}^*(\mathbf{p}+|\mathbf{q}-)],\end{aligned}\tag{4.8a}$$

$$\begin{aligned}[b_{-p}^*,\mathbf{H}]=&-E_p b_{-p}^*-e\sum_q\phi(\mathbf{p}-\mathbf{q})[a_q(\mathbf{p}-|\mathbf{q}+)\\ &+b_{-q}^*(\mathbf{p}-|\mathbf{q}-)].\end{aligned}\tag{4.8b}$$

Equations (4.7) are therefore linear relations in the amplitudes

$$\begin{aligned}u(\mathbf{p})&=(\Phi_0,a_p\Phi_{1,n}),\\ v(\mathbf{p})&=(\Phi_0,b_{-p}^*\Phi_{1,n}),\end{aligned}\tag{4.9}$$

and read

$$\begin{aligned}&(E_p-\mathcal{E}_{1n})u(\mathbf{p})+e\sum\phi(\mathbf{p}-\mathbf{q})\\ &\qquad\times\{(\mathbf{p}+|\mathbf{q}+)u(\mathbf{q})+(\mathbf{p}+|\mathbf{q}-)v/\mathbf{q})\}=0,\\ &(E_p+\mathcal{E}_{1n})v(\mathbf{p})+e\sum\phi(\mathbf{p}-\mathbf{q})\\ &\qquad\times\{(\mathbf{p}-|\mathbf{q}-)v(\mathbf{q})+(\mathbf{p}-|\mathbf{q}+)u(\mathbf{q})\}=0.\end{aligned}\tag{4.10}$$

The terms in $v(\mathbf{p})$ are present only because of the polarization of the vacuum. For the naive vacuum

$v(\mathbf{p})=0$. Thus the presence of the two equations in (4.10) and the coupling between them arises in the field theoretic interpretations as a consequence of vacuum polarization.

But (4.10) are just the wave equations in the free-particle representation, and identical with (2.55) for bosons, (3.36) for fermions. There is one difference, however: By assumption, $\mathcal{E}$ is positive in this case, so that only the positive solutions of (4.10) are to be associated with the problem at hand. The negative solutions to (4.10) arise from a different problem, namely the eigenvalue problem for negative charge states $\mathbf{\Phi}_{-1,n}$:

$$\mathbf{H}\mathbf{\Phi}_{-1,n}=\mathcal{E}_{-1,n}\mathbf{\Phi}_{-1,n}.$$

In analogy to the method used for the positive charge states, we consider the equations,

$$\begin{aligned}(\mathbf{\Phi}_{-1,n},[a_p,\mathbf{H}]\mathbf{\Phi}_0)&=-\mathcal{E}_{-1,n}(\mathbf{\Phi}_{-1,n},a_p\mathbf{\Phi}_0),\\ (\mathbf{\Phi}_{-1,n},[b_{-p}{}^*,\mathbf{H}]\mathbf{\Phi}_0)&=-\mathcal{E}_{-1,n}(\mathbf{\Phi}_{-1,n},b_{-p}{}^*\mathbf{\Phi}_0).\end{aligned}\quad(4.9a)$$

Working out the commutator gives an equation *identical* to (4.10), for the amplitudes

$$\begin{aligned}u'(\mathbf{p})&=(\mathbf{\Phi}_{-1},a_p\mathbf{\Phi}_0),\\ v'(\mathbf{p})&=(\mathbf{\Phi}_{-1},b_{-p}{}^*\mathbf{\Phi}_0),\end{aligned}\quad(4.11)$$

except that $E=\mathcal{E}_{1,n}$ is now replaced by $E'=-\mathcal{E}_{-1,n}<0$. This again demonstrates the proper interpretation of the negative eigenstates of (4.10) as the solutions to the charge-conjugate problem.

It would have been possible to define directly a set of amplitudes u_c, v_c for the charge-conjugate problem. For bosons, for example, this would simply be achieved by defining

$$\begin{aligned}u_c(\mathbf{p})&=(\mathbf{\Phi}_0,b_p\mathbf{\Phi}_{-1}),\\ v_c(\mathbf{p})&=(\mathbf{\Phi}_0,a_{-p}{}^*\mathbf{\Phi}_{-1}).\end{aligned}\quad(4.11a)$$

The equations for u_c and v_c are then like (4.10), for $E=\mathcal{E}_{-1,n}>0$, but with the sign of the charge e reversed.

By the nature of the procedure that leads to (4.10), the eigenvalues of this equation define the one-particle energy spectrum of the field Hamiltonian $\mathbf{H}$. As long as the external field is sufficiently weak (e.g., $Ze^2/\hbar c<\frac{1}{2}$ for bosons, $Ze^2/\hbar c<1$ for fermions in an external Coulomb field), the regular solutions of (4.10) form a complete, orthonormal set and, therefore, give the complete one-particle spectrum of $\mathbf{H}$. If the field is too strong difficulties arise: The admissible solutions are no longer automatically orthogonal.[23] The physics behind this situation is the breakdown of the validity of the external field approximation. An external field builds up around its source a polarization charge density

$$\begin{aligned}\rho_{\rm pol}&=(e/2)\langle 0|\,\mathbf{\Psi}^*(\mathbf{x})\mathbf{\Psi}(\mathbf{x})-\mathbf{\Psi}(\mathbf{x})\mathbf{\Psi}^*(\mathbf{x})\,|0\rangle,\quad\text{(fermions)},\\ \rho_{\rm pol}&=(e/2)\langle 0|\,\mathbf{\Psi}^*\tau_3\mathbf{\Psi}-\mathbf{\Psi}(\mathbf{\Psi}^*\tau_3)\,|0\rangle.\quad\text{(bosons)}.\end{aligned}$$

[23] K. M. Case, Phys. Rev. **80**, 797 (1950).

The external field approximation *neglects* the electromagnetic interaction between this polarization density and the particle. This polarization effect *prevents*[24] the actual occurrence of fields exceeding a critical strength. Thus, from a physical point of view, no actual difficulty arises; only our approximate treatment of the situation becomes then inadequate. Still, a one-particle equation, of type (4.10) exists, but the original external field ϕ is supplemented by a screening field $\phi_{\rm pol}$. This system of equations has well-defined solutions which form a complete set.

Returning now to the original situation (4.5), the orthogonality relations for the solutions of (4.10) follow directly from the definitions of the amplitudes $u(p)$ and $v(p)$. From the commutation relations (4.5a,b) we get relations of the type

$$\begin{aligned}\textstyle\sum_n \langle 0|\,a_p\,|1n\rangle\langle 1n\,|\,a_q{}^*\,|0\rangle\\ -\textstyle\sum_n \langle 0|\,a_q{}^*\,|-1,n\rangle\langle -1,n\,|\,a_p\,|0\rangle=\delta_{pq}.\end{aligned}$$

This equation holds exactly only if all states with unit charge are included in the sum. This is more than the single particle as defined earlier; that is, the sum must also include states which have a particle of a given unit charge plus any number of pairs of positively and negatively charged particles. However, these cannot exist at large distances from the external field since, from (4.10), the energy momentum relation for a single particle holds there. These states must, therefore, contribute to the polarization charge density and are important when strong external fields are present. Their effect may be taken into account by a suitable modification in the external field.[25]

Now with this reservation in mind we employ the definitions of the amplitudes u and v. Equations (4.9) and (4.11) may be written as

$$\begin{aligned}\textstyle\sum_n\sum_\sigma \sigma u_{\sigma n}(\mathbf{p})u_{\sigma n}{}^*(\mathbf{q})&=\delta_{pq}\\ \textstyle\sum_n\sum_\sigma \sigma v_{\sigma n}(\mathbf{p})v_{\sigma n}{}^*(\mathbf{q})&=-\delta_{pq}\end{aligned}\quad\text{(bosons)},\quad(4.12a)\dagger$$

and

$$\begin{aligned}\textstyle\sum_{\sigma,n} u_{\sigma n}(\mathbf{p})u^*{}_{\sigma n}(\mathbf{q})&=\delta_{pq}\\ \textstyle\sum_{\sigma,n} v_{\sigma n}(\mathbf{p})v^*{}_{\sigma n}(\mathbf{q})&=\delta_{pq}\end{aligned}\quad\text{(fermions)}.\quad(4.12b)$$

Similarly from the zero commutator of a_p and b_q we have the equations

$$\begin{aligned}\textstyle\sum_{u',v} \sigma u_{\sigma n}(\mathbf{p})v^*{}_{\sigma n}(\mathbf{q})&=0\\ \textstyle\sum_{\sigma,n} \sigma v_{\sigma n}(\mathbf{p})v^*{}_{\sigma n}(\mathbf{q})&=0\end{aligned}\quad\text{for bosons.}$$

[24] J. M. Jauch and F. Rohrlich (Addison-Wesley Press, Cambridge, 1955), pp. 311–312.

[25] The effect of this field is particularly important in the case of μ-mesic atoms where $\phi_{\rm pol}$ is the consequence of the polarization of the electron-positron vacuum by the field of the nucleus.

† In these equations $u(\sigma=+1)$ stands for the u's defined by (4.11), $u(\sigma=-1)$ for the u's defined by (4.11a).

These are obviously "completeness-relations." As such, they imply the orthogonality relations[26]

$$\sum_q [u_{\sigma n}^*(\mathbf{q})u_{\tau m}(\mathbf{q})-v_{\sigma n}^*(\mathbf{q})v_{\tau m}(\mathbf{q})]=\sigma\delta_{\sigma\tau}\delta_{nm} \quad \text{for bosons,} \quad (4.13a)$$

and

$$\sum_q [u_{\sigma n}^*(\mathbf{q})u_{\tau m}(\mathbf{q})+v_{\sigma n}^*(\mathbf{q})v_{\tau m}(\mathbf{q})]=\delta_{\sigma\tau}\delta_{mn} \quad \text{for fermions.} \quad (4.13b)$$

With this, we have rederived all the basic elements of the wave-mechanical description.

B. Neutral Particles

We now proceed to a discussion of neutral particles, which by definition do not interact with the electromagnetic field. However, the relativistic doubling of states still occurs, and we must rename these two degrees of freedom so as to apply equally to both charged and neutral particles. For example, in the scalar case, if Ψ describes the particle, then $\tau_1\Psi^*$ describes the antiparticle. The process of going from Ψ to $\tau_1\Psi^*$ is called particle-antiparticle conjugation rather than charge conjugation. In the fermion case, the neutrino and antineutrino form a possible particle-antiparticle combination.

The convenience of such a terminology depends upon whether or not interactions with other fields distinguishes between particles and antiparticles as, for example, the electromagnetic field distinguishes between positive and negative charge. If the interaction does distinguish then the particle and antiparticle have an analog of the charge associated with each, e.g., the nucleonic charge[27] recently introduced. On the other hand, if the interactions with other fields is identical for both particle and antiparticle it becomes possible to make an "abbreviation" of the theory so that only one degree of freedom enters as was first done for fermions by Majorana.[28]

Obviously it becomes important to classify interactions according to their behavior under particle-antiparticle conjugation. We restrict the discussion to the boson case. First let us assume that the interaction does *not* remove the particle-antiparticle degeneracy. Then both Ψ and $\tau_1\Psi^*$ are solutions of the Schroedinger equation so that

$$\hbar i\frac{\partial}{\partial t}(\tau_1\Psi^*)=H(\tau_1\Psi^*).$$

Comparing this to the equation satisfied by Ψ^* we immediately see that H must satisfy the equation,

$$\tau_1 H\tau_1=-H^*. \quad (4.13)$$

To this must be added the condition of Hermiticity [(2.32), (2.33)],

$$\tau_3\bar{H}\tau_3=H,$$

as well as the condition of Lorentz covariance. We find that only operators of the form

$$H=A\tau_3+B(\tau_3+i\tau_2),$$

where A is a real constant while B can be a real scalar space-time function. Therefore, the form of H must be

$$H=(\tau_3+i\tau_2)(\phi^2/2m)+\tau_3 mc^2+(\tau_3+i\tau_2)gS(\mathbf{x}). \quad (4.14)$$

The corresponding modified Klein-Gordon equation is

$$(\Box^2-\kappa^2)\psi-(\kappa g/\hbar c)S\psi=0. \quad (4.15)$$

If the interactions are not of the form (4.14), then one may define an analog of the charge and take over the development given in Sec. 2 without any essential changes.

We now discuss the situation given by Hamiltonian (4.14). We develop a Majorana abbreviation of the theory in which the particle and the antiparticle are identified. The Schroedinger equation corresponding to (4.14) is

$$\begin{aligned} i\hbar(\partial\varphi/\partial t)&=(p^2/2m)(\varphi+\chi)+mc^2\varphi+gS(\varphi+\chi),\\ i\hbar(\partial\chi/\partial t)&=(-p^2/2m)(\varphi+\chi)-mc^2\chi-gS(\varphi+\chi). \end{aligned} \quad (4.16)$$

From the preceding discussion these equations are invariant under the substitutions,

$$\begin{aligned} \varphi&\rightarrow\pm\chi^*,\\ \chi&\rightarrow\pm\varphi^*, \end{aligned} \quad (4.17)$$

so that the charge conjugate equation is identical with the original. In other words, the complete set of eigenfunctions for (4.16) is given by

$$\begin{pmatrix}\varphi_n(\mathbf{x})\\ \chi_n(\mathbf{x})\end{pmatrix}e^{-(i/\hbar)E_n t} \text{ and } \begin{pmatrix}\chi^*_n(\mathbf{x})\\ \varphi^*_n(\mathbf{x})\end{pmatrix}e^{(i/\hbar)E_n t} \quad E_n>0 \text{ only.}$$

These wave functions can be normalized

$$\int d^3x(\varphi^*_n\varphi_n-\chi^*_n\chi_n)=1, \quad (4.18)$$

but it is not clear how this furnishes a basis for interpretation since the integrand and the associated "current density" are not associated with charge distributions. To obtain an interpretation it is necessary to go to quantum field theory since here the interpretation may be obtained from the field intensity and energy operators for the field, as is done in electromagnetic theory.

There are two Majorana abbreviations possible, one for which $\Psi=\Psi_c$ and one for which $\Psi=-\Psi_c$; or, in

[26] These orthogonality relations may also be derived directly from the expression for the charge operator (4.6) and the equation $\langle n\sigma|Q|m\tau\rangle=e\delta_{\sigma\tau}\delta_{mn}$.

[27] J. Schwinger, Phys. Rev. **104**, 1164 (1956).

[28] E. Majorana, Nuovo cimento **14**, 171 (1937). W. Furry, Phys. Rev. **54**, 56 (1938). See also discussion by W. Pauli, reference 19.

other words, one which has an eigenfunction of the particle-antiparticle conjugation operator has the eigenvalue $(+1)$, the other (-1). The π^0 meson is known (in virtue of its decay into two gamma rays) to be even under charge conjugation so that we shall consider here only the first of the above two cases. We write

$$\Psi=\sum[c_p\Psi^{(+)}(\mathbf{p})+c^*_{-p}\Psi^{(-)}(\mathbf{p})]\frac{\exp(i\mathbf{p}\cdot\mathbf{x}/\hbar)}{\sqrt{2}}. \quad (4.19)$$

We note

$$\tau_1\Psi^*=\Psi^c=\Psi. \quad (4.20)$$

[The wave function which is odd under charge conjugation is given by taking the difference of the two terms in the sum in (4.19) and dividing by i.] Comparing this with the charged particle case we see that the dynamically independent operators a_p and b_p have been replaced by a single operator c_p. Equivalently, in the charged particle case Ψ and Ψ^c were independent operators. In the above abbreviated theory they are the same.

We now examine the consequences of (4.19). The charge operator becomes

$$Q=\int d^3x(\Psi^*\tau_3\Psi+\Psi(\Psi^*\tau_3))\equiv 0, \quad (4.21)$$

as expected. For Hamiltonian (4.14), the field Hamiltonian

$$\mathbf{H}=\int d^3x\Psi^*\tau_3H\Psi$$

is

$$\mathbf{H}=\sum_p E_pc^*_pc_p+g\sum_{p,q}S(\mathbf{p}-\mathbf{q})\frac{mc^2}{(E_pE_q)^{\frac{1}{2}}}$$
$$\times[c^*_pc_q+\tfrac{1}{2}c_pc_{-q}+\tfrac{1}{2}c^*_pc^*_{-q}]. \quad (4.22)$$

Again the "naive" vacuum is not an eigenstate of $\mathbf{H}$. Contrary, however, to the charged particle case, there is no way of labeling the eigenstates of $\mathbf{H}$ by a charge quantum number and indeed the question of the number of particles associated with a given eigenstate is meaningful only in the limit of weak interactions. We can only use the eigenvalues of the Hamiltonian to label the states. The physical vacuum state Φ_0 is taken to be the lowest energy state with energy zero (by adjustment of the energy scale). The one-particle state, Φ_1, is then the first excited state of the system, etc., assuming, of course, that the interactions with other fields are not too strong. We may then define the one-particle wave function by

$$u(\mathbf{p})=(\Phi_0,c_p\Phi_1),$$
$$v(\mathbf{p})=(\Phi_0,c_{-p}{}^*\Phi_1). \quad (4.23)$$

As may be seen in the weak energy limit and principally because we are dealing with a system with only one degree of freedom, no other independent matrix elements can be formed. Utilizing expressions analogous to (4.8), one may immediately derive the equation satisfied by u and v. We find

$$Eu=E_pu+g\int d^3qS(\mathbf{p}-\mathbf{q})\frac{mc^2}{(E_pE_q)^{\frac{1}{2}}}[u(\mathbf{q})+v(\mathbf{q})],$$
$$Ev=-E_pv-g\int d^3qS(\mathbf{p}-\mathbf{q})\frac{mc^2}{(E_pE_q)^{\frac{1}{2}}}[u(\mathbf{q})+v(\mathbf{q})]. \quad (4.24)$$

These are just the Schroedinger equations in momentum space following from Hamiltonian (4.14).

We turn now to the physical interpretation of the above equation. In the case of charged particles the discussion depends on the charge and current density operators. For example, the orthonormal conditions which are so necessary for the statistical interpretation can be obtained directly from the conservation of charge.[26] In the neutral case we use instead the field intensity $\mathbf{S}$ and energy density which also satisfy a continuity equation. This is similar to the procedure followed in electromagnetic theory, where the physical interpretation of the Maxwell equations, particularly for scattering problems, is made in terms of the Poynting vector $c/4\pi(\mathbf{E}\times\mathbf{H})$. For the present interaction and in terms of ψ of Sec. 2

$$S_k=-(\hbar^2/m)[(\partial\psi^*/\partial t)(\partial\psi/\partial x_k)+(\partial\psi^*/\partial x_k)(\partial\psi/\partial t)].$$

This expression is exact even when the interaction S is included. In terms of Ψ,

$$S_k=-\frac{\hbar c^2}{2i}\left[\frac{\partial\Psi^*}{\partial x_k}(\tau_3+i\tau_2)\Psi-\Psi^*(\tau_3-i\tau_2)\frac{\partial\Psi}{\partial x_k}\right]$$
$$=\frac{c^2}{2}[(p_k\Psi^*)(\tau_3+i\tau_2)\Psi+\Psi^*(\tau_3-i\tau_2)p_k\Psi].$$

Dropping a divergence term

$$S_k=c^2[\Psi^*\tau_3p_k\Psi]. \quad (4.25)$$

Now going over to quantum field theory, we insert expansion (4.19), and integrating over all space we obtain

$$\mathcal{S}_k=\int S_kd^3x=c^2\sum p_k[c^*_pc_p-c_{-p}c^*_{-p}]. \quad (4.26)$$

If we now take the one-particle expectation value, we obtain

$$\langle 1|\mathcal{S}_k|1\rangle=c^2\int p_k[u^*(\mathbf{p})u(\mathbf{p})-v^*(\mathbf{p})v(\mathbf{p})]d^3p. \quad (4.27)$$

We need only this expression for the calculation of

scattering cross section since from it we can obtain both the incident intensity and the scattered power. Combining (4.27) with the orthogonality relation which follows from (4.24)

$$\int (u^*_n u_m - v^*_n v_m) d^3p = \delta_{nm}, \tag{4.28}$$

we may interpret

$$u_n^*(\mathbf{p}) u_n(\mathbf{p}) - v_n^*(\mathbf{p}) v_n(\mathbf{p})$$

as being the density in momentum space of the intensity. The corresponding expression in ordinary space is

$$\varphi_n^*(\mathbf{x}) \varphi_n(\mathbf{x}) - \chi_n^*(\mathbf{x}) \chi_n(\mathbf{x}).$$

This completes our discussion of the neutral particle.

C. Wave-Mechanical Description of Boson of Isotope Spin 1

It is now easily seen how one can construct a wave-mechanical description of bosons with isotopic spin 1, having the charge states 1, 0, −1. We construct a six-component wave function

$$\Theta = \begin{bmatrix} \Psi \\ \Psi_0 \\ \Psi_c \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{bmatrix}. \tag{4.29}$$

Here Ψ is the wave function of a charged boson of two components (2.16), Ψ_c is the charge conjugate to Ψ, and Ψ_0 is the wave function of a neutral particle.

We may now introduce not only a scalar interaction $S(x)$ common to all amplitudes in (4.20), but also a charge-exchange interaction

$$\sum_\alpha T_\alpha V_\alpha(\mathbf{x}). \tag{4.30}$$

T_α being the isotopic spin matrices for isotopic spin one, and V_α an isotopic vector. V_α, describes a change of charge of the scatterer. The wave equation for Θ is

$$i\hbar \frac{\partial \Theta}{\partial t} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c} T_3 \mathbf{A}\right)^2 (\tau_3 + i\tau_2)\Theta + mc^2 \tau_3 \Theta + eT_3 \phi \Theta + \tfrac{1}{2}(g_S S + g_V (\mathbf{T}\cdot\mathbf{V}))(\tau_3 + i\tau_2)\Theta. \tag{4.31}$$

The isotopic spin operator $\mathbf{T}$ is given by the usual angular momentum matrices for angular momentum 1:

$$T_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad T_2 = \frac{i}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{4.32}$$

From the method by which (4.31) is obtained, it is clear that only the positive energy solutions of this problem refer to the situation for which this equation is intended.

The negative solutions of (4.31) describe the scattering by a "mirror sytem" (say, by the mirror nucleus of the original scattering nucleus).

A note on the solution of (4.31) for a scattering problem may be helpful. We have endeavored to put these equations into Hamiltonian form as a bridge to the discussion of their physical content. This does not mean that this is the form most suitable for solving the equation. The well-known problem of the eigenfunctions of a charged boson in a Coulomb field, both for bound and continuum states, is an illustration of the point: It is then much more appropriate to solve the one-component equation in the Klein-Gordon form directly. The same holds for the scattering of mesons of isotopic spin 1 by a nuclear potential, as exemplified by (4.31). The Klein-Gordon equation associated with (4.31) is

$$(E - e\phi)^2 \psi_\lambda = \left(\mathbf{p} - \frac{\lambda e}{c}\right)^2 \psi_\lambda + mc^2 \psi_\lambda + g_S S \psi_\lambda + g_V \sum_{\lambda'} \langle \lambda | \mathbf{T} | \lambda' \rangle \cdot \mathbf{V} \psi_\lambda \tag{4.33}$$

with

$$\psi_\lambda = \frac{1}{\sqrt{2}}(\varphi_\lambda + \chi_\lambda)$$

according to (2.13).

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