PHYS4101 Thermal Physics
(Fall 2021)

Midterm Exam #2
Tuesday 2 Nov 2021

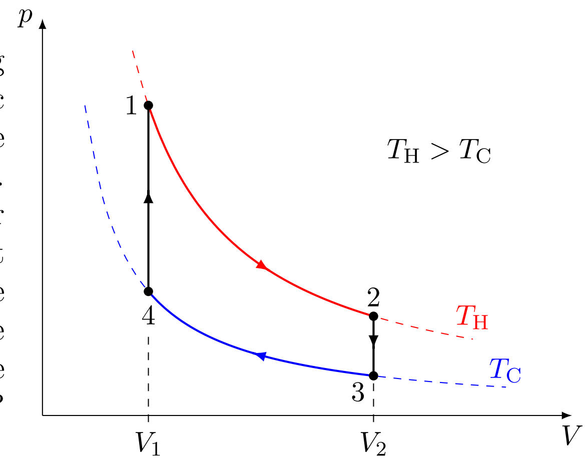
There are **four questions** and you are to work all of them. You are welcome to use your textbook, notes, or any other resources, but you may not communicate with another human. Of course, if you have questions, you are encouraged to ask the person proctoring the exam.

The four problems will be equally weighted but *they are not of equal difficulty.* If you are stuck on one, move on to another and come back if you have time.

Please start each problem on a new page in your exam booklet.

Good luck!

(1) The figure is a PV diagram for a Stirling Engine, operating with one mole of a monatomic ideal gas. In terms of T_C, T_H, V_1, and V_2, find the work done and heat exchanged on each of the legs. Use this to find the work done by the engine over one complete cycle. Then find the absorbed heat Q_H and the expelled heat Q_C. Finally, calculate the efficiency of the Stirling Engine and compare it to that of the Carnot cycle. If T_H and T_C are fixed, what can you do to optimize the efficiency?



(2) Find an expression for the pressure $P = P(T)$ that describes the shape of the *liquid-gas* phase transition boundary on a pressure vs temperature phase diagram. Use the assumptions that (i) you are in a region where the latent heat L (per mole) is constant, (ii) the volume of the liquid is negligible compared to that of the gas, and (iii) the ideal gas law is valid. You can leave your answer in terms of some arbitrary constant.

(3) We showed in class that the probably of finding an atom in a particular *state* while in thermal equilibrium with a reservoir at temperature T is given by $\mathcal{P}(E) = (1/Z)\exp(-E/kT)$ where E is the energy of the state, and the partition function Z comes from summing the Boltzmann factors over all states. If you interpret the degeneracy of any particular energy *level* as the multiplicity Ω, then show that the probability of finding the atom in the *energy level* with energy E is $\mathcal{P}(E) = (1/Z)\exp(-F/kT)$ where F is the Helmholtz Free Energy.

(4) Show that $v_{\text{rms}} = \sqrt{3kT/m}$ by finding the average of v^2 over the Maxwell Distribution of speeds in an ideal gas. There's an integral to do, which you can look up, but I will give extra credit to anyone who shows how it can be done with a "Gaussian integral" trick.

Solutions

(1) Legs $1 \to 2$ and $3 \to 4$ are isotherms so $W = Q = RT\log(V_2/V_1)$ for each of them. No work is done for legs $2 \to 3$ and $4 \to 1$ so $Q = \Delta U = (3/2)R\Delta T$ for each of them. Heat enters on $4 \to 1$ and $1 \to 2$ and leaves on $2 \to 3$ and $3 \to 4$ so, the work done over one cycle is $W = R(T_H - T_C)\log(V_2/V_1)$ and, with everything positive,

$$W_{1\to2} = RT_H \log\frac{V_2}{V_1} = Q_{1\to2} \qquad W_{3\to4} = RT_C \log\frac{V_2}{V_1} = Q_{3\to4} \qquad W_{2\to3} = 0 = W_{4\to1}$$

$$Q_H = RT_H \log\frac{V_2}{V_1} + \frac{3}{2}R(T_H - T_C) \qquad Q_C = RT_C \log\frac{V_2}{V_1} + \frac{3}{2}R(T_H - T_C)$$

$$e = \frac{Q_H - Q_C}{Q_H} = \frac{(T_H - T_C)\log(V_2/V_1)}{T_H\log(V_2/V_1) + 3(T_H - T_C)/2} = \frac{e_{\text{Carnot}}}{1 + 3e_{\text{Carnot}}/2\log(V_2/V_1)}$$

which is, as expected, less than for the Carnot cycle. With T_H and T_C fixed, the only way you can make the second term in the dominator smaller is to increase the "compression ratio" V_2/V_1 as much as possible. But you won't win quickly, because of the logarithm.

(2) See Problem 5.35 in the textbook. We want use the Clausius-Clapeyron relation as a differential equation, that is $dP/dT = L/T\Delta V$. Putting $\Delta V = V = RT/P$ for the ideal gas gives $dP/P = LdT/RT^2$ so $\log P = -L/RT + \text{constant}'$ so $P(T) = \text{constant} \times e^{-L/RT}$.

(3) See Problem 6.2 in the textbook. This is a very simple problem. If $g \to \Omega$ is the degeneracy, then the probability to find the system in the *energy level* E is

$$\mathcal{P}(E) = \frac{1}{Z}\Omega e^{-E/kT} = \frac{1}{Z}e^{S/k}e^{-E/kT} = \frac{1}{Z}e^{-(E-TS)/kT} = \frac{1}{Z}e^{-F/kT}$$

(4) We need to average v^2 with the probability distribution $\mathcal{D}(v)$ from (6.50). The integral involves a fourth power of v times an exponential, which we get using the second derivative version of the Gaussian integral trick. So, we have

$$\begin{aligned}
v_{\text{rms}}^2 &= \int_0^\infty v^2\mathcal{D}(v)dv = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi \int_0^\infty v^4 e^{-mv^2/2kT}dv \\
&= \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi \left(\frac{2kT}{m}\right)^{5/2} \int_0^\infty x^4 e^{-x^2}dx = \frac{8kT}{m}\frac{1}{\sqrt{\pi}}\int_0^\infty x^4 e^{-x^2}dx \\
\int_0^\infty x^4 e^{-x^2}dx &= \frac{d}{da^2}\int_0^\infty e^{-ax^2}dx\bigg|_{a=1} = \frac{d}{da^2}\frac{\sqrt{\pi}}{2}a^{-1/2}\bigg|_{a=1} = \frac{\sqrt{\pi}}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{3\sqrt{\pi}}{8}
\end{aligned}$$

$$\text{Therefore} \qquad v_{\text{rms}}^2 = \frac{3kT}{m} \qquad \text{and} \qquad v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$