Name: ____________________

PHYS3701 Intro to Quantum Mechanics I   Quiz #1   26 Jan 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

The state vector for a particular quantum mechanical system is given by

$$|\psi\rangle = c|{+\mathbf{z}}\rangle + i\sqrt{\frac{1}{3}}|{-\mathbf{z}}\rangle$$

where c is a complex number.

(i) What is the most general form for c so that $|\psi\rangle$ is normalized?

(ii) What is the probability that a measurement of S_z gives the value $+\hbar/2$?

(iii) **Extra credit:** *Assuming that c is a real number*, what is the probability that a measurement of S_x gives the value $+\hbar/2$?

PHYS3701 Intro to Quantum Mechanics Quiz #1 Solution

The state vector for a particular quantum mechanical system is given by

$$|\psi\rangle = c|+\mathbf{z}\rangle + i\sqrt{\frac{1}{3}}|-\mathbf{z}\rangle$$

where c is a complex number.

(i) What is the most general form for c so that $|\psi\rangle$ is normalized?

(ii) What is the probability that a measurement of S_z gives the value $+\hbar/2$?

(iii) **Extra credit:** *Assuming that c is a real number*, what is the probability that a measurement of S_x gives the value $+\hbar/2$?

(i) For $\langle\psi|\psi\rangle = 1$ we must have

$$c^*c + \left(-i\sqrt{\frac{1}{3}}\right)\left(i\sqrt{\frac{1}{3}}\right) = |c|^2 + \frac{1}{3} = 1 \qquad \text{so} \qquad |c|^2 = \frac{2}{3} \qquad \text{and} \qquad c = e^{i\delta}\sqrt{\frac{2}{3}}$$

where δ is an arbitrary real number.

(ii) The probability is $|\langle+\mathbf{z}|\psi\rangle|^2 = |c|^2 = 2/3$.

(iii) The probability is

$$\begin{aligned}
|\langle+\mathbf{x}|\psi\rangle|^2 &= \left|\left[\frac{1}{\sqrt{2}}\langle+\mathbf{z}| + \frac{1}{\sqrt{2}}\langle-\mathbf{z}|\right]|\psi\rangle\right|^2 \\
&= \frac{1}{2}\left|c + i\sqrt{\frac{1}{3}}\right|^2 = \frac{1}{2}\left|\sqrt{\frac{2}{3}} + i\sqrt{\frac{1}{3}}\right|^2 = \frac{1}{6}(\sqrt{2}-i)(\sqrt{2}+i) = \frac{1}{2}
\end{aligned}$$

Name: ______________________________

PHYS3701 Intro to Quantum Mechanics I Quiz #2 2 Feb 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

The state vector for a particular quantum mechanical system is given by

$$|\psi\rangle = \sqrt{\frac{2}{3}}|+\mathbf{z}\rangle + i\sqrt{\frac{1}{3}}|-\mathbf{z}\rangle$$

Calculate the expectation $\langle S_z\rangle$ for this state. You can do this from the original definition of $\langle S_z\rangle$ as an average of a bunch of measurements; or directly from the state and the appropriate operator; or using matrix manipulations. I will give extra credit for those of you who do it more than one way to check your answer.

PHYS3701 Intro to Quantum Mechanics Quiz #2 <u>Solution</u>

The state vector for a particular quantum mechanical system is given by

$$|\psi\rangle = \sqrt{\frac{2}{3}}|+\mathbf{z}\rangle + i\sqrt{\frac{1}{3}}|-\mathbf{z}\rangle$$

Calculate the expectation $\langle S_z\rangle$ for this state. You can do this from the original definition of $\langle S_z\rangle$ as an average of a bunch of measurements; or directly from the state and the appropriate operator; or using matrix manipulations. I will give extra credit for those of you who do it more than one way to check your answer.

First recognize that $\langle S_z\rangle = \langle\psi|\hat{J}_z|\psi\rangle$. Using matrices we have

$$\left[\sqrt{\frac{2}{3}}, -i\sqrt{\frac{1}{3}}\right]\left[\begin{array}{cc}\hbar/2 & 0\\ 0 & -\hbar/2\end{array}\right]\left[\begin{array}{c}\sqrt{2/3}\\ i\sqrt{1/3}\end{array}\right] = \frac{\hbar}{2}\left[\sqrt{\frac{2}{3}}, -i\sqrt{\frac{1}{3}}\right]\left[\begin{array}{c}\sqrt{2/3}\\ -i\sqrt{1/3}\end{array}\right] = \frac{\hbar}{2}\left[\frac{2}{3}-\frac{1}{3}\right] = \frac{\hbar}{6}$$

Using the operator and states directly, we have

$$\begin{aligned}
& \left[\sqrt{\frac{2}{3}}\langle+\mathbf{z}| - i\sqrt{\frac{1}{3}}\langle-\mathbf{z}|\right]\hat{J}_z\left[\sqrt{\frac{2}{3}}|+\mathbf{z}\rangle + i\sqrt{\frac{1}{3}}|-\mathbf{z}\rangle\right]\\
= & \left[\sqrt{\frac{2}{3}}\langle+\mathbf{z}| - i\sqrt{\frac{1}{3}}\langle-\mathbf{z}|\right]\left[\sqrt{\frac{2}{3}}\left(\frac{\hbar}{2}\right)|+\mathbf{z}\rangle + i\sqrt{\frac{1}{3}}\left(-\frac{\hbar}{2}\right)|-\mathbf{z}\rangle\right]\\
= & \frac{2}{3}\left(\frac{\hbar}{2}\right) + \frac{1}{3}\left(-\frac{\hbar}{2}\right) = \frac{\hbar}{6}
\end{aligned}$$

Going back to our original definition as an average measurement, we just have

$$\left|\sqrt{\frac{2}{3}}\right|^2\left(\frac{\hbar}{2}\right) + \left|i\sqrt{\frac{1}{3}}\right|^2\left(-\frac{\hbar}{2}\right) = \frac{2}{3}\left(\frac{\hbar}{2}\right) + \frac{1}{3}\left(-\frac{\hbar}{2}\right) = \frac{\hbar}{6}$$

Name: ______________________________

PHYS3701 Intro to Quantum Mechanics I Quiz #3 9 Feb 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

In a notation where angular momentum eigenstates are written as usual in the form $|j, m\rangle$, calculate the two matrix elements

$$\langle 1, 1|\hat{J}_x|1, 0\rangle \qquad \text{and} \qquad \langle 1, -1|\hat{J}_y|1, 0\rangle$$

You may find it useful to consider first the results of $\hat{J}_\pm|1, 0\rangle$.

PHYS3701 Intro to Quantum Mechanics Quiz #3 <u>Solution</u>

In a notation where angular momentum eigenstates are written as usual in the form $|j,m\rangle$, calculate the two matrix elements

$$\langle 1,1|\hat{J}_x|1,0\rangle \qquad \text{and} \qquad \langle 1,-1|\hat{J}_y|1,0\rangle$$

You may find it useful to consider first the results of $\hat{J}_\pm|1,0\rangle$.

The key relationships are

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \qquad \text{and} \qquad \hat{J}_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}\hbar\,|j,m\pm 1\rangle$$

Note that $j(j+1)-m(m\pm 1) = j^2 - m^2 + j \mp m = (j\mp m)(j\pm m+1)$ which is how this factor is written in other textbooks, including MQM3e. Now

$$\begin{aligned}
\hat{J}_+|1,0\rangle &= \sqrt{1(2)-0}\hbar\,|1,1\rangle = \sqrt{2}\hbar\,|1,1\rangle \\
\text{and} \qquad \hat{J}_-|1,0\rangle &= \sqrt{1(2)-0}\hbar\,|1,1\rangle = \sqrt{2}\hbar\,|1,-1\rangle \\
\text{so} \qquad \langle 1,1|\hat{J}_+|1,0\rangle &= \sqrt{2}\hbar \\
\text{and} \qquad \langle 1,1|\hat{J}_-|1,0\rangle &= 0 \\
\text{so} \qquad \langle 1,1|\hat{J}_x|1,0\rangle &= \frac{1}{2}\left[\langle 1,1|\hat{J}_+|1,0\rangle + \langle 1,1|\hat{J}_-|1,0\rangle\right] = \frac{1}{\sqrt{2}}\hbar \\
\text{and} \qquad \langle 1,-1|\hat{J}_y|1,0\rangle &= \frac{1}{2i}\left[\langle 1,-1|\hat{J}_+|1,0\rangle - \langle 1,-1|\hat{J}_-|1,0\rangle\right] = \frac{i}{\sqrt{2}}\hbar
\end{aligned}$$

Name: ______________________________

PHYS3701 Intro to Quantum Mechanics I Quiz #4 16 Feb 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

In a notation where angular momentum eigenstates are written as usual in the form $|j, m\rangle$, calculate the four matrix elements

$$\langle 1,0|\hat{J}_x|1,0\rangle \qquad \text{and} \qquad \langle 1,-1|\hat{J}_x|1,0\rangle \qquad \text{and} \qquad \langle 1,0|\hat{J}_y|1,0\rangle \qquad \text{and} \qquad \langle 1,1|\hat{J}_y|1,0\rangle$$

I suggest that you first find the matrix elements of $\hat{J}_+$ and $\hat{J}_-$ for the same four pairs of states.

PHYS3701 Intro to Quantum Mechanics Quiz #4 Solution

In a notation where angular momentum eigenstates are written as usual in the form $|j, m\rangle$, calculate the four matrix elements

$$\langle 1,0|\hat{J}_x|1,0\rangle \quad \text{and} \quad \langle 1,-1|\hat{J}_x|1,0\rangle \quad \text{and} \quad \langle 1,0|\hat{J}_y|1,0\rangle \quad \text{and} \quad \langle 1,1|\hat{J}_y|1,0\rangle$$

I suggest that you first find the matrix elements of $\hat{J}_+$ and $\hat{J}_-$ for the same four pairs of states.

The key relationships are

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \quad \text{and} \quad \hat{J}_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\hbar\,|j, m\pm 1\rangle$$

Note that $j(j+1) - m(m \pm 1) = j^2 - m^2 + j \mp m = (j \mp m)(j \pm m + 1)$ which is how this factor is written in other textbooks, including MQM3e. We have

$$\begin{aligned} \hat{J}_+|1,0\rangle &= \sqrt{1(2)-0}\hbar\,|1,1\rangle = \sqrt{2}\hbar\,|1,1\rangle \\ \text{and} \qquad \hat{J}_-|1,0\rangle &= \sqrt{1(2)-0}\hbar\,|1,1\rangle = \sqrt{2}\hbar\,|1,-1\rangle \end{aligned}$$

It is therefore obvious that

$$\langle 1,0|\hat{J}_+|1,0\rangle = \langle 1,0|\hat{J}_-|1,0\rangle = 0$$

because $|1,\pm 1\rangle$ are both orthogonal to $|1,0\rangle$. It is also clear that

$$\begin{aligned} \langle 1,-1|\hat{J}_+|1,0\rangle &= 0 \\ \langle 1,-1|\hat{J}_-|1,0\rangle &= \sqrt{2}\hbar \\ \langle 1,1|\hat{J}_+|1,0\rangle &= \sqrt{2}\hbar \\ \langle 1,1|\hat{J}_-|1,0\rangle &= 0 \end{aligned}$$

Now to make it very clear how to go from $\hat{J}_\pm$ to $\hat{J}_x$ and $\hat{J}_y$, write things down in terms of numbers. That is

$$\begin{aligned} \langle\alpha|\hat{J}_+|\beta\rangle &= \langle\alpha|\hat{J}_x|\beta\rangle + i\langle\alpha|\hat{J}_y|\beta\rangle \\ \text{and} \qquad \langle\alpha|\hat{J}_-|\beta\rangle &= \langle\alpha|\hat{J}_x|\beta\rangle - i\langle\alpha|\hat{J}_y|\beta\rangle \\ \text{so} \qquad \langle\alpha|\hat{J}_x|\beta\rangle &= \frac{1}{2}\left[\langle\alpha|\hat{J}_+|\beta\rangle + \langle\alpha|\hat{J}_-|\beta\rangle\right] \\ \text{and} \qquad \langle\alpha|\hat{J}_y|\beta\rangle &= \frac{1}{2i}\left[\langle\alpha|\hat{J}_+|\beta\rangle - \langle\alpha|\hat{J}_-|\beta\rangle\right] \end{aligned}$$

so the results for the remaining two matrix elements are

$$\begin{aligned} \langle 1,-1|\hat{J}_x|1,0\rangle &= \frac{1}{2}\left[0 + \sqrt{2}\hbar\right] = \frac{1}{\sqrt{2}}\hbar \\ \text{and} \qquad \langle 1,1|\hat{J}_y|1,0\rangle &= \frac{1}{2i}\left[\sqrt{2}\hbar - 0\right] = -\frac{i}{\sqrt{2}}\hbar \end{aligned}$$

Name: ________________________

PHYS3701 Intro to Quantum Mechanics I Quiz #5 2 Mar 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

A particle of mass m sits in a one-dimensional infinite well such that $V(x) = 0$ for $0 \le x \le L$. The particle cannot be found outside the well. At time $t = 0$, the wave function is

$$\psi(x, t=0) = \frac{1}{2}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} + \frac{i}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L} - \frac{1}{2}\sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}$$

for $0 \le x \le L$ and zero elsewhere. Find (i) the energy expectation value $\langle E \rangle$ and (ii) the probability that the particle is found in its lowest energy state, each as a function of time. (If you want to say that they are independent of time, explain your reasoning.)
Note: You do not need to do any lengthy calculations to find the answer.

PHYS3701 Intro to Quantum Mechanics Quiz #5 Solution

A particle of mass m sits in a one-dimensional infinite well such that $V(x) = 0$ for $0 \le x \le L$. The particle cannot be found outside the well. At time $t = 0$, the wave function is

$$\psi(x, t = 0) = \frac{1}{2}\sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} + \frac{i}{\sqrt{2}}\sqrt{\frac{2}{L}} \sin\frac{2\pi x}{L} - \frac{1}{2}\sqrt{\frac{2}{L}} \sin\frac{3\pi x}{L}$$

for $0 \le x \le L$ and zero elsewhere. Find (i) the energy expectation value $\langle E \rangle$ and (ii) the probability that the particle is found in its lowest energy state, each as a function of time. (If you want to say that they are independent of time, explain your reasoning.)
<u>Note</u>: You do not need to do any lengthy calculations to find the answer.

The wave function is a linear combination of the $n = 1$, $n = 2$, and $n = 3$ eigenstates. The Hamiltonian is time-independent and, of course, commutes with itself, so the expectation value of energy must be constant. We have

$$\langle E \rangle = \frac{1}{4}\frac{\hbar^2\pi^2}{2mL^2}1^2 + \frac{1}{2}\frac{\hbar^2\pi^2}{2mL^2}2^2 + \frac{1}{4}\frac{\hbar^2\pi^2}{2mL^2}3^2 = \frac{18}{4}\frac{\hbar^2\pi^2}{2mL^2} = \frac{9}{4}\frac{\hbar^2\pi^2}{mL^2}$$

The projection onto any eigenstate for the time-dependent wave function will give something whose time dependence is $e^{-iE_nt/\hbar}$, so this gives no time-dependence to the magnitude of the square. Therefore the probability is time-independent as well. The lowest energy state ($n = 1$) has linear coefficient $1/2$, so

$$\text{Prob for } n = 1 \text{ is } \frac{1}{4}$$

Name: ______________________________

PHYS3701 Intro to Quantum Mechanics I Quiz #6 9 Mar 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

A particle of mass m is bound by a harmonic oscillator potential energy $V(x) = m\omega^2x^2/2$. Find the expectation value $\langle V(x)\rangle$ for the potential energy if the system is in the energy eigenstate $|n\rangle$. Also find the expectation value of the kinetic energy $p_x^2/2m$; this is a very simple calculation given $\langle V(x)\rangle$.

PHYS3701 Intro to Quantum Mechanics Quiz #6 Solution

A particle of mass m is bound by a harmonic oscillator potential energy $V(x) = m\omega^2 x^2/2$. Find the expectation value $\langle V(x)\rangle$ for the potential energy if the system is in the energy eigenstate $|n\rangle$. Also find the expectation value of the kinetic energy $p_x^2/2m$; this is a very simple calculation given $\langle V(x)\rangle$.

From (7.11), we have

$$\begin{aligned}
\langle V(x)\rangle &= \frac{1}{2}m\omega^2\langle n|\hat{x}^2|n\rangle = \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}\langle n|(\hat{a}+\hat{a}^\dagger)^2|n\rangle = \frac{\hbar\omega}{4}\langle n|(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})|n\rangle \\
&= \frac{\hbar\omega}{4}\left(\sqrt{n+1}\sqrt{n+1}+\sqrt{n}\sqrt{n}\right) = \frac{\hbar\omega}{4}(2n+1) = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right) \\
&= \frac{1}{2}E_n = \frac{1}{2}\langle E\rangle
\end{aligned}$$

Clearly this means that $\langle p_x^2/2m\rangle = E_n/2$ as well, but we can easily do this out from scratch. This time we make use of (7.12) and write

$$\left\langle\frac{p_x^2}{2m}\right\rangle = \frac{1}{2m}\langle n|\hat{p}_x^2|n\rangle = -\frac{1}{2m}\frac{m\omega\hbar}{2}\langle n|(\hat{a}-\hat{a}^\dagger)^2|n\rangle = \frac{\hbar\omega}{4}\langle n|(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})|n\rangle$$

which is the same expression we evaluated above.

Name: ____________________

PHYS3701 Intro to Quantum Mechanics I Quiz #7 16 Mar 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

The three components of the vector angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ are

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \qquad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_z$$

Use the commutation relations between position and momenta to prove that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$.

PHYS3701 Intro to Quantum Mechanics Quiz #7 Solution

The three components of the vector angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ are

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \qquad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_z$$

Use the commutation relations between position and momenta to prove that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$.

$$\begin{aligned}
[\hat{L}_x, \hat{L}_y] &= \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x \\
&= (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) - (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y) \\
&= +\hat{y}\hat{p}_z\hat{z}\hat{p}_x - \hat{y}\hat{p}_z\hat{x}\hat{p}_z - \hat{z}\hat{p}_y\hat{z}\hat{p}_x + \hat{z}\hat{p}_y\hat{x}\hat{p}_z \\
&\quad -\hat{z}\hat{p}_x\hat{y}\hat{p}_z + \hat{x}\hat{p}_z\hat{y}\hat{p}_z + \hat{z}\hat{p}_x\hat{z}\hat{p}_y - \hat{x}\hat{p}_z\hat{z}\hat{p}_y \\
&= \hat{y}(\hat{p}_z\hat{z} - \hat{z}\hat{p}_z)\hat{p}_x + 0 + 0 + \hat{x}(\hat{z}\hat{p}_z - \hat{p}_z\hat{z})\hat{p}_y \\
&= \hat{y}(-i\hbar)\hat{p}_x + \hat{x}(+i\hbar)\hat{p}_y \\
&= i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) \\
&= i\hbar\hat{L}_z
\end{aligned}$$

Name: ____________________

PHYS3701 Intro to Quantum Mechanics I Quiz #8 23 Mar 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

Calculate the expectation value of z^2 for the $n = 2$ state of a one-electron atom with atomic number $Z = 2$ and angular momentum quantum numbers $l = 1$ and $m = 1$. Express your answer as a numerical factor times the square of the Bohr radius. You will find it useful that

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \qquad \text{and} \qquad \int_0^\pi \sin\theta d\theta \, f(\cos\theta) = \int_{-1}^{1} d\mu \, f(\mu)$$

PHYS3701 Intro to Quantum Mechanics Quiz #8 Solution

Calculate the expectation value of z^2 for the $n = 2$ state of a one-electron atom with atomic number $Z = 2$ and angular momentum quantum numbers $l = 1$ and $m = 1$. Express your answer as a numerical factor times the square of the Bohr radius. You will find it useful that

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \qquad \text{and} \qquad \int_0^\pi \sin\theta d\theta\, f(\cos\theta) = \int_{-1}^1 d\mu\, f(\mu)$$

$$\begin{aligned}
\langle z^2\rangle &= \langle 211|\hat{z}^2|211\rangle = \int d^3r \langle 211|\hat{z}^2|\mathbf{r}\rangle\langle\mathbf{r}||211\rangle = \int d^3r R_{21}^*(r)Y_{11}^*(\theta,\phi)z^2 R_{21}(r)Y_{11}(\theta,\phi) \\
&= \int d^3r\, R_{21}^*(r)Y_{11}^*(\theta,\phi)\, r^2\cos^2\theta\, R_{21}(r)Y_{11}(\theta,\phi) \\
&= \int_0^\infty r^2 dr\, r^2 R_{21}^*(r)R_{21}(r)\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left(-\sqrt{\frac{3}{8\pi}}e^{-i\phi}\sin\theta\right)\cos^2\theta\left(-\sqrt{\frac{3}{8\pi}}e^{i\phi}\sin\theta\right) \\
&= \frac{3}{8\pi}2\pi\int_0^\infty r^2 dr\, r^2 R_{21}^*(r)R_{21}(r)\int_0^\pi \sin\theta d\theta\, \sin^2\theta\cos^2\theta \\
&= \frac{3}{4}\int_0^\infty r^2 dr\, r^2 R_{21}^*(r)R_{21}(r)\int_{-1}^1 d\mu(1-\mu^2)\mu^2 \\
&= \frac{3}{4}\int_0^\infty r^2 dr\, r^2 R_{21}^*(r)R_{21}(r)\left[2\left(\frac{1}{3}-\frac{1}{5}\right)\right] = \frac{3}{4}\frac{4}{15}\int_0^\infty r^2 dr\, r^2 R_{21}^*(r)R_{21}(r) \\
&= \frac{1}{5}\int_0^\infty r^4 \frac{1}{3}\left(\frac{Z}{2a_0}\right)^3\frac{Z^2r^2}{a_0^2}e^{-Zr/a_0}dr \\
&= \frac{1}{5}\frac{1}{3}\frac{1}{8}\frac{Z^5}{a_0^5}\int_0^\infty r^6 e^{-Zr/a_0}dr \\
&= \frac{1}{5}\frac{1}{3}\frac{1}{8}\frac{Z^5}{a_0^5}6!\left(\frac{a_0}{Z}\right)^7 = \frac{6\cdot 5\cdot 4\cdot 3\cdot 4\cdot 1}{5\cdot 3\cdot 8}\left(\frac{a_0}{Z}\right)^2 = 12\left(\frac{a_0}{Z}\right)^2 = 3a_0^2
\end{aligned}$$

Name: ____________________

PHYS3701 Intro to Quantum Mechanics I Quiz #9 30 Mar 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

Calculate the expectation value of x^2 for the $n = 2$ state of a one-electron atom with atomic number $Z = 2$ and angular momentum quantum numbers $l = 1$ and $m = 1$. Express your answer as a numerical factor times the square of the Bohr radius. You will find it useful that

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} , \qquad \cos^2\phi = \frac{1+\cos 2\phi}{2} , \text{ and } \qquad \int_0^\pi \sin\theta d\theta\, f(\cos\theta) = \int_{-1}^1 d\mu\, f(\mu)$$

PHYS3701 Intro to Quantum Mechanics Quiz #9 Solution

Calculate the expectation value of x^2 for the $n = 2$ state of a one-electron atom with atomic number $Z = 2$ and angular momentum quantum numbers $l = 1$ and $m = 1$. Express your answer as a numerical factor times the square of the Bohr radius. You will find it useful that

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \,, \qquad \cos^2\phi = \frac{1+\cos 2\phi}{2} \,, \text{ and } \qquad \int_0^\pi \sin\theta d\theta\, f(\cos\theta) = \int_{-1}^1 d\mu\, f(\mu)$$

$$\begin{aligned}
\langle x^2\rangle &= \langle 211|\hat{x}^2|211\rangle = \int d^3r \langle 211|\hat{x}^2|\mathbf{r}\rangle\langle\mathbf{r}||211\rangle = \int d^3r R_{21}^*(r) Y_{11}^*(\theta,\phi) x^2 R_{21}(r) Y_{11}(\theta,\phi) \\
&= \int d^3r\, R_{21}^*(r) Y_{11}^*(\theta,\phi)\, r^2 \sin^2\theta \cos^2\phi\, R_{21}(r) Y_{11}(\theta,\phi) \\
&= \int_0^\infty r^2 dr\, r^2 R_{21}^*(r) R_{21}(r) \int_0^{2\pi} d\phi\, \cos^2\phi \\
&\quad \times \int_0^\pi \sin\theta d\theta \left(-\sqrt{\frac{3}{8\pi}} e^{-i\phi}\sin\theta\right) \sin^2\theta \left(-\sqrt{\frac{3}{8\pi}} e^{i\phi}\sin\theta\right) \\
&= \frac{3}{8\pi}\pi \int_0^\infty r^2 dr\, r^2 R_{21}^*(r) R_{21}(r) \int_0^\pi \sin\theta d\theta\, \sin^4\theta \\
&= \frac{3}{8} \int_0^\infty r^2 dr\, r^2 R_{21}^*(r) R_{21}(r) \int_{-1}^1 d\mu (1-\mu^2)^2 \\
&= \frac{3}{8} \int_0^\infty r^2 dr\, r^2 R_{21}^*(r) R_{21}(r) \int_{-1}^1 d\mu (1-2\mu^2+\mu^4) \\
&= \frac{3}{8} \int_0^\infty r^2 dr\, r^2 R_{21}^*(r) R_{21}(r) \left[2\left(1-\frac{2}{3}+\frac{1}{5}\right)\right] = \frac{3}{8}\frac{12}{15} \int_0^\infty r^2 dr\, r^2 R_{21}^*(r) R_{21}(r) \\
&= \frac{3}{10} \int_0^\infty r^4 \frac{1}{3}\left(\frac{Z}{2a_0}\right)^3 \frac{Z^2 r^2}{a_0^2} e^{-Zr/a_0} dr \\
&= \frac{3}{10}\frac{1}{3}\frac{1}{8}\frac{Z^5}{a_0^5} \int_0^\infty r^6 e^{-Zr/a_0} dr \\
&= \frac{3}{10}\frac{1}{3}\frac{1}{8}\frac{Z^5}{a_0^5} 6! \left(\frac{a_0}{Z}\right)^7 = \frac{6\cdot5\cdot4\cdot3\cdot3\cdot4\cdot1}{5\cdot2\cdot3\cdot8}\left(\frac{a_0}{Z}\right)^2 = 18\left(\frac{a_0}{Z}\right)^2 = \frac{9}{2}a_0^2
\end{aligned}$$

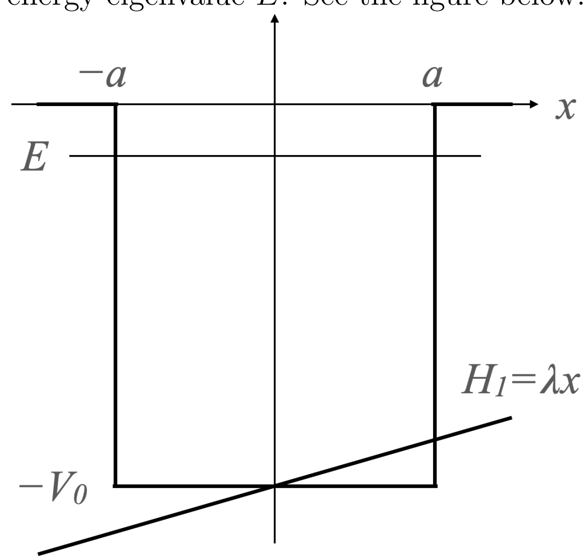
Name: ______________________________

PHYS3701 Intro to Quantum Mechanics I Quiz #10 6 Apr 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

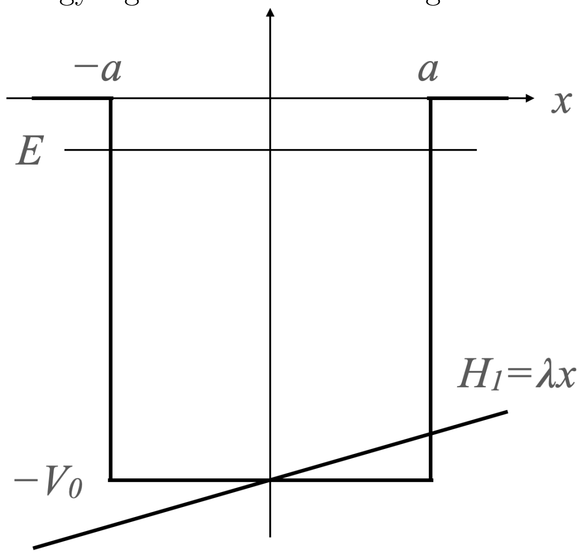
A particle of mass m is confined to a finite height symmetric square well in one dimension x. The well has a depth $-V_0$ and extends from $-a \leq x \leq a$. For this potential energy, there is one bound state with energy eigenvalue E. See the figure below.



A perturbation $\hat{H}_1(\hat{x}) = \lambda\hat{x}$, where λ is a positive constant, is then added to this potential energy function. Find the *first order* shift in the energy E due to this perturbation.

PHYS3701 Intro to Quantum Mechanics Quiz #10 Solution

A particle of mass m is confined to a finite height symmetric square well in one dimension x. The well has a depth $-V_0$ and extends from $-a \le x \le a$. For this potential energy, there is one bound state with energy eigenvalue E. See the figure below.



A perturbation $\hat{H}_1(\hat{x}) = \lambda\hat{x}$, where λ is a positive constant, is then added to this potential energy function. Find the *first order* shift in the energy E due to this perturbation.

The first order energy shift is zero. The perturbation is an odd function, and the potential energy for $\hat{H}_0$ is symmetric, so the zeroth order wave functions $\langle x|E\rangle$ are either even or odd, and the integral from $-\infty$ to ∞ must give zero. To be very explicit about it,

$$
\begin{aligned}
E^{(1)} &= \langle E|\hat{H}_1|E\rangle \\
&= \int_{-\infty}^{\infty} dx \langle E|\hat{H}_1|x\rangle\langle x|E\rangle \\
&= \int_{-\infty}^{\infty} dx \langle E|x\rangle \lambda x \langle x|E\rangle \\
&= \lambda \int_{-\infty}^{\infty} dx\, x\, |\langle x|E\rangle|^2 \\
&= \lambda \int_{0}^{\infty} dx\, x\, |\langle x|E\rangle|^2 + \lambda \int_{-\infty}^{0} dx\, x\, |\langle x|E\rangle|^2 \\
&= \lambda \int_{0}^{\infty} dx\, x\, |\langle x|E\rangle|^2 + \lambda \int_{\infty}^{0} dy\, y\, |\langle y|E\rangle|^2 \\
&= \lambda \int_{0}^{\infty} dx\, x\, |\langle x|E\rangle|^2 - \lambda \int_{0}^{\infty} dy\, y\, |\langle y|E\rangle|^2 \\
&= \lambda \int_{0}^{\infty} dx\, x\, |\langle x|E\rangle|^2 - \lambda \int_{0}^{\infty} dx\, x\, |\langle x|E\rangle|^2 = 0
\end{aligned}
$$

where we substituted $y = -x$ and made use of the fact that $\langle \pm x|E\rangle = \pm\langle x|E\rangle$, and also, in the last step, that the integration variable is a dummy, so we changed y back to x.

Name: ______________________________

PHYS3701 Intro to Quantum Mechanics I Quiz #11 13 Apr 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

A particle of mass m is in a one-dimensional "box" where the potential $V(x) = 0$ for $-a/2 \le x \le +a/2$, and $V(x) = \infty$ otherwise. Find the first order energy shift in the ground state due to the perturbation $H_1 = \lambda x^2$. You will find the following integral useful:

$$\int x^2 \cos^2(kx)\, dx = \frac{x\cos(2kx)}{4k^2} + \frac{(2k^2x^2 - 1)\sin(2kx)}{8k^3} + \frac{x^3}{6} + C$$

PHYS3701 Intro to Quantum Mechanics Quiz #11 Solution

A particle of mass m is in a one-dimensional "box" where the potential $V(x) = 0$ for $-a/2 \le x \le +a/2$, and $V(x) = \infty$ otherwise. Find the first order energy shift in the ground state due to the perturbation $H_1 = \lambda x^2$. You will find the following integral useful:

$$\int x^2 \cos^2(kx)\,dx = \frac{x\cos(2kx)}{4k^2} + \frac{(2k^2x^2-1)\sin(2kx)}{8k^3} + \frac{x^3}{6} + C$$

See Section 6.9 for the zeroth order solution. The $n = 1$ wave function (6.108) is $\psi_1(x) = \sqrt{2/a}\,\cos(\pi x/a)$ for $-a/2 \le x \le +a/2$ and zero otherwise, so the first order energy shift is

$$\begin{aligned}
E_1^{(1)} &= \langle \phi_1^{(0)}|\hat{H}_1|\phi_1^{(0)}\rangle = \int_{-\infty}^{\infty} dx\,\langle \phi_1^{(0)}|x\rangle \lambda x^2 \langle x|\phi_1^{(0)}\rangle = \int_{-\infty}^{\infty} dx\,\psi_1^*(x)\,\lambda x^2\,\psi_1(x) \\
&= \lambda\frac{2}{a}\int_{-a/2}^{+a/2} x^2\,\cos^2\left(\frac{\pi x}{a}\right)dx \\
&= \lambda\frac{2}{a}\left[\frac{a^2}{4\pi^2}x\cos\left(\frac{2\pi x}{a}\right) + \left(\frac{ax^2}{4\pi} - \frac{a^3}{8\pi^3}\right)\sin\left(\frac{2\pi x}{a}\right) + \frac{x^3}{6}\right]_{-a/2}^{+a/2} \\
&= \lambda\frac{2}{a}2\left[-\frac{a^3}{8\pi^2} + \frac{a^3}{48}\right] \\
&= \lambda a^2\left[\frac{1}{12} - \frac{1}{2\pi^2}\right]
\end{aligned}$$

Name: ______________________________

PHYS3701 Intro to Quantum Mechanics I   Quiz #12   20 Apr 2021

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary.

A Hermitian operator $\hat{A}$ in a system with three eigenstates $|1\rangle$, $|2\rangle$, and $|3\rangle$ is given by

$$\hat{A} = a|1\rangle\langle 1| + ib\left(|2\rangle\langle 3| - |3\rangle\langle 2|\right)$$

where a and b are real numbers. Find the eigenvalues of $\hat{A}$ in terms of a and b.

Extra Credit: Find the three eigenvectors of $\hat{A}$ in terms of $|1\rangle$, $|2\rangle$, and $|3\rangle$.

PHYS3701 Intro to Quantum Mechanics Quiz #12 Solution

A Hermitian operator $\hat{A}$ in a system with three eigenstates $|1\rangle$, $|2\rangle$, and $|3\rangle$ is given by

$$\hat{A} = a|1\rangle\langle 1| + ib\left(|2\rangle\langle 3| - |3\rangle\langle 2|\right)$$

where a and b are real numbers. Find the eigenvalues of $\hat{A}$ in terms of a and b.

Extra Credit: Find the three eigenvectors of $\hat{A}$ in terms of $|1\rangle$, $|2\rangle$, and $|3\rangle$.

Representing $\hat{A}$ in the $|1\rangle$, $|2\rangle$, $|3\rangle$ basis, the eigenvalue equation is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & ib \\ 0 & -ib & 0 \end{pmatrix}\begin{pmatrix} \langle 1|\lambda\rangle \\ \langle 2|\lambda\rangle \\ \langle 3|\lambda\rangle \end{pmatrix} = \lambda\begin{pmatrix} \langle 1|\lambda\rangle \\ \langle 2|\lambda\rangle \\ \langle 3|\lambda\rangle \end{pmatrix}$$

We therefore solve for λ using

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & -\lambda & ib \\ 0 & -ib & -\lambda \end{vmatrix} = (a-\lambda)\lambda^2 - b^2(a-\lambda) = (a-\lambda)(\lambda^2-b^2) = 0$$

so $\lambda = a$ and $\lambda = \pm b$. Note that the determinant can also be reduced simply as

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & -\lambda & ib \\ 0 & -ib & -\lambda \end{vmatrix} = (a-\lambda)\begin{vmatrix} -\lambda & ib \\ -ib & -\lambda \end{vmatrix} = (a-\lambda)(\lambda^2-b^2)$$

For $\lambda = a$, find $\langle 2|a\rangle = \langle 3|a\rangle = 0$ so $|a\rangle = |1\rangle$. For $\lambda = \pm b$ we need to solve

$$\begin{pmatrix} 0 & ib \\ -ib & 0 \end{pmatrix}\begin{pmatrix} \langle 2|\pm b\rangle \\ \langle 3|\pm b\rangle \end{pmatrix} = \pm b\begin{pmatrix} \langle 2|\pm b\rangle \\ \langle 3|\pm b\rangle \end{pmatrix}$$

and the bottom equation gives

$$-ib\langle 2|\pm b\rangle = \pm b\langle 3|\pm b\rangle \qquad \text{so} \qquad \langle 3|\pm b\rangle = \mp i\langle 2|\pm b\rangle$$

and the normalized eigenvectors are

$$|\pm b\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle \mp i|3\rangle\right)$$

We can go back to the original form for $\hat{A}$ and check. Clearly $\hat{A}|1\rangle = a|1\rangle$. Also

$$\begin{aligned} \hat{A}|\pm b\rangle &= ib\left(|2\rangle\langle 3| - |3\rangle\langle 2|\right)\frac{1}{\sqrt{2}}\left(|2\rangle \mp i|3\rangle\right) \\ &= \pm\frac{1}{\sqrt{2}}b|2\rangle - \frac{1}{\sqrt{2}}ib|3\rangle = \pm b\left(\frac{1}{\sqrt{2}}|2\rangle \mp \frac{i}{\sqrt{2}}|3\rangle\right) = \pm b|\pm b\rangle \end{aligned}$$