PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #9

Due at 5pm to the **Grader** on Thursday 18 Mar 2021

(1) Prove the relation between the square of the orbital angular momentum operator $\hat{\mathbf{L}}^2$ and the 3D position and momentum operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ that we used in class, namely

$$\hat{\mathbf{L}}^2 = \hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = \hat{L}_i \hat{L}_i = \hat{\mathbf{r}}^2 \hat{\mathbf{p}}^2 - (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2 + i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}$$

Make use of the definition $\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$ (Remember: We are using the Einstein summation convention!), the commutation relation $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, and the very useful theorem

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

I'm assigning this problem mostly because I want you to appreciate the power of this notation. It's tricky, though, so here are some tips. First, note that, for two vectors $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A} \cdot \mathbf{B} = A_i B_i = A_j B_j = \delta_{ij} A_i B_j$ where any two repeated indices are summed, so they are dummy indices. Remember that all three $\hat{x}_i$ commute with each other, as do all $\hat{p}_i$, whereas $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ tells you how to "flip" the order of position and momentum. You also have $\delta_{ij}\delta_{ik} = \delta_{kj}$, and similar examples. Can you convince yourself that $\delta_{kk} = 3$?

So use the expression above for $\varepsilon_{ijk}\varepsilon_{ilm}$, and look at what momenta and positions are "paired" by the Kronecker delta to guide yourself towards the result you want to obtain. You can get to the final answer in eight or ten lines of algebra.

(2) The wave function for a particular particle has the form

$$\psi(\mathbf{r}) = (x + y + z) f(r)$$

where $f(r)$ is some arbitrary function of the radial spherical coordinate. Express $\psi(\mathbf{r})$ in terms of spherical harmonics, tabulated in many places including Equations (9.151), (9.152), and (9.153) in your textbook, and then determine the following:

a. What are the possible results of a measurement of $\hat{\mathbf{L}}^2$, and what are the probabilities for getting these results?

b. What are the possible results of a measurement of $\hat{L}_z$, and what are the probabilities for getting these results?