PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #1

Due at 5pm to the **Grader** on Thursday 21 Jan 2021

(1) A beam of silver atoms is created by heating a vapor in an oven to 1227°C, and selecting atoms with a velocity close to the mean of the thermal distribution. The beam moves through a 1 m long magnetic field with a vertical gradient 10 T/m, and impinges a screen 1 m downstream of the end of the magnet. Assuming the silver atom has spin-1/2 with a magnetic moment of one Bohr magneton, find the separation distance in mm of the two states on the screen.

(2) A solid uniform sphere of mass m has a charge q uniformly distributed over its surface. The sphere spins on its axis with angular momentum $\mathbf{L}$. Calculate its magnetic moment $\boldsymbol{\mu} = g(q/2mc)\mathbf{L}$ and determine the (dimensionless) constant g. (If you work the problem in SI units, you will not get the factor of c in the denominator.)

Solutions HW #1

(1) See the MATHEMATICA solutions notebook. We find a separation of 3.0 mm.

(2) The magnetic moment of a ring of radius r and current I is $\pi r^2 I$ (in SI units), in the direction normal to the plane of the ring. Since the current is generated by the rotating charged surface, it is clear that $\boldsymbol{\mu}$ is in the direction of $\mathbf{L}$.

We will integrate over the polar angle $0 \leq \theta \leq \pi$ to get the magnetic moment. The radius of a ring around the surface of the sphere at angle θ is $r = R\sin\theta$. The surface area of the ring that subtends an angle $d\theta$ is $ds = 2\pi r(Rd\theta) = 2\pi R^2 \sin\theta\, d\theta$ which we can check with

$$\int_0^\pi ds = 2\pi R^2 \int_0^\pi \sin\theta\, d\theta = -2\pi R^2 \cos\theta\big|_0^\pi = 4\pi R^2$$

The charge in this ring is $dq = q\, ds/4\pi R^2 = (q/2)\sin\theta d\theta$. The current dI in the ring is the charge divided by the time T for one revolution, so $dI = dq/T = dq(\omega/2\pi)$, where ω is the angular velocity. The individual magnetic moments add up, so

$$\mu = \int d\mu = \int \pi r^2 dI = \frac{\omega}{2}\int r^2 dq = \frac{\omega}{2}\frac{q}{2}R^2 \int_0^\pi \sin^3\theta\, d\theta = \frac{\omega}{2}\frac{q}{2}R^2 \int_{-1}^{1}(1-t^2)dt = \frac{\omega q R^2}{3}$$

The moment of inertia of a solid uniform sphere with mass m and radius R is $(2/5)mR^2$, so its angular momentum is $L = (2/5)mR^2\omega$ and

$$\mu = \frac{q}{3}\frac{5}{2}\frac{1}{m}L = g\frac{q}{2m}L \qquad \text{where} \qquad g = \frac{5}{3}$$

which agrees with the statement of Problem 1.2 in Townsend. Note that in CGS units, the definition of the magnetic dipole moment contains a factor of $1/c$.

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Homework Assignment #2

Due at 5pm to the <u>**Grader**</u> on Thursday 28 Jan 2021

(1) Consider a quantum state $|\psi(t)\rangle$ which changes with time t as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{e^{i\omega t}}{\sqrt{2}}|-\mathbf{z}\rangle$$

where ω is a real number. Interpreting the expectation value as the average of a large number of measurements, calculate $\langle S_z \rangle$ and $\langle S_y \rangle$ as a function of time. What is happening, physically? Can you guess what $\langle S_x \rangle$ turns out to be without actually calculating it?

(2) Using the $|+\mathbf{y}\rangle$ and $|-\mathbf{y}\rangle$ states as a basis, find the representation (as column vectors) for the states $|+\mathbf{x}\rangle$ and $|-\mathbf{x}\rangle$.

Solutions HW #2

(1) We need first to calculate $|\langle\pm\mathbf{z}|\psi(t)\rangle|^2$ and $|\langle\pm\mathbf{y}|\psi(t)\rangle|^2$:

$$\begin{aligned}
\langle+\mathbf{z}|\psi(t)\rangle &= \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \langle-\mathbf{z}|\psi(t)\rangle = \frac{e^{i\omega t}}{\sqrt{2}} \qquad \text{so} \qquad |\langle\pm\mathbf{z}|\psi(t)\rangle|^2 = \frac{1}{2} \\
\text{and} \qquad \langle+\mathbf{y}|\psi(t)\rangle &= \left[\frac{1}{\sqrt{2}}\langle+\mathbf{z}| - \frac{i}{\sqrt{2}}\langle-\mathbf{z}|\right]|\psi(t)\rangle = \frac{1}{2}\left(1 - ie^{i\omega t}\right) = \frac{1}{2}\left(1 - i\cos\omega t + \sin\omega t\right) \\
\langle-\mathbf{y}|\psi(t)\rangle &= \left[\frac{1}{\sqrt{2}}\langle+\mathbf{z}| + \frac{i}{\sqrt{2}}\langle-\mathbf{z}|\right]|\psi(t)\rangle = \frac{1}{2}\left(1 + ie^{i\omega t}\right) = \frac{1}{2}\left(1 + i\cos\omega t - \sin\omega t\right) \\
\text{so} \qquad |\langle\pm\mathbf{y}|\psi(t)\rangle|^2 &= \frac{1}{2}(1 \pm \sin\omega t)
\end{aligned}$$

The expectation values are straightforward from here:

$$\begin{aligned}
\langle S_z\rangle &= \frac{1}{2}\left(\frac{\hbar}{2}\right) + \frac{1}{2}\left(-\frac{\hbar}{2}\right) = 0 \\
\langle S_y\rangle &= \frac{1}{2}(1+\sin\omega t)\left(\frac{\hbar}{2}\right) + \frac{1}{2}(1-\sin\omega t)\left(-\frac{\hbar}{2}\right) = \frac{\hbar}{2}\sin\omega t
\end{aligned}$$

The spin precesses about the z-axis. Note that at $t = 0$, $|\psi(t)\rangle = |+\mathbf{x}\rangle$, so of course $\langle S_y\rangle = 0$ at $t = 0$. It is pretty obvious, in fact, that we must find $\langle S_x\rangle = (\hbar/2)\cos\omega t$.

(2) We have $|\psi\rangle = |+\mathbf{y}\rangle\langle+\mathbf{y}|\psi\rangle + |-\mathbf{y}\rangle\langle-\mathbf{y}|\psi\rangle$, so calculate $\langle\pm\mathbf{y}|+\mathbf{x}\rangle$ and $\langle\pm\mathbf{y}|-\mathbf{x}\rangle$:

$$\begin{aligned}
\langle\pm\mathbf{y}|+\mathbf{x}\rangle &= \frac{1}{2}\left[\langle+\mathbf{z}| \mp i\langle-\mathbf{z}|\right]\left[|+\mathbf{z}\rangle + |-\mathbf{z}\rangle\right] = \frac{1}{2}(1 \mp i) \\
\langle\pm\mathbf{y}|-\mathbf{x}\rangle &= \frac{1}{2}\left[\langle+\mathbf{z}| \mp i\langle-\mathbf{z}|\right]\left[|+\mathbf{z}\rangle - |-\mathbf{z}\rangle\right] = \frac{1}{2}(1 \pm i)
\end{aligned}$$

Therefore, in the $|\pm\mathbf{y}\rangle$ basis,

$$|+\mathbf{x}\rangle \doteq \begin{bmatrix} \frac{1-i}{2} \\ \frac{1+i}{2} \end{bmatrix} \qquad \text{and} \qquad |-\mathbf{x}\rangle \doteq \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix}$$

Note that this is Problem 2.4 in the textbook. This answer agrees with the solutions manual.

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Homework Assignment #3

Due at 5pm to the <u>**Grader**</u> on Thursday 4 Feb 2021

(1) Find the resulting ket, as an expansion in the $|\pm\mathbf{z}\rangle$ basis, that results from $\hat{R}(\alpha\mathbf{j})|+\mathbf{z}\rangle$ where $\hat{R}(\alpha\mathbf{j})$ is the operator which effects a counter-clockwise rotation through an angle α about the y-axis. Check to see if your answer makes sense for $\alpha = 0$, $\alpha = \pi/2$, and $\alpha = \pi$. Recall (2.42) and (2.43a) in the textbook.

Hints. Base your form of $\hat{R}(\alpha\mathbf{j})|+\mathbf{z}\rangle$ on (2.32) in the textbook. You will need to evaluate $\hat{J}_y|\pm\mathbf{z}\rangle$ to work this out, and that is most easily done by writing $|+\mathbf{z}\rangle$ in terms of the $|\pm\mathbf{y}\rangle$.

(2) Prove the following properties of commutators:

(i) $\left[\hat{A}, \hat{B} + \hat{C}\right] = \left[\hat{A}, \hat{B}\right] + \left[\hat{A}, \hat{C}\right]$

(ii) $\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$

(ii) $\left[\hat{A}, \hat{B}\hat{C}\right] = \hat{B}\left[\hat{A}, \hat{C}\right] + \left[\hat{A}, \hat{B}\right]\hat{C}$

Solutions HW #3

(1) This is Problem 2.6 in the textbook. The operator $\hat{R}(\alpha\mathbf{j}) = e^{-iJ_y\alpha/\hbar}$, so we want to write $|+\mathbf{z}\rangle$ in terms of the $|\pm\mathbf{y}\rangle$. Do this from (2.16) or (2.17a) and (2.19b), that is

$$|\pm\mathbf{y}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle \pm \frac{i}{\sqrt{2}}|-\mathbf{z}\rangle \qquad \text{so} \qquad |+\mathbf{z}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{y}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{y}\rangle$$

$$\begin{aligned}
\hat{R}(\alpha\mathbf{j})|+\mathbf{z}\rangle &= \hat{R}(\alpha\mathbf{j})\left[\frac{1}{\sqrt{2}}|+\mathbf{y}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{y}\rangle\right] = \frac{e^{-i\alpha/2}}{\sqrt{2}}|+\mathbf{y}\rangle + \frac{e^{+i\alpha/2}}{\sqrt{2}}|-\mathbf{y}\rangle \\
\hat{R}(0\mathbf{j})|+\mathbf{z}\rangle &= \frac{1}{\sqrt{2}}|+\mathbf{y}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{y}\rangle = |+\mathbf{z}\rangle \\
\hat{R}\left(\frac{\pi}{2}\mathbf{j}\right)|+\mathbf{z}\rangle &= \frac{e^{-i\pi/4}}{\sqrt{2}}|+\mathbf{y}\rangle + \frac{e^{+i\pi/4}}{\sqrt{2}}|-\mathbf{y}\rangle = \frac{1-i}{2}|+\mathbf{y}\rangle + \frac{1+i}{2}|-\mathbf{y}\rangle \\
&= \frac{1}{2}\frac{2}{\sqrt{2}}|+\mathbf{z}\rangle - \frac{i}{2}\frac{2i}{\sqrt{2}}|-\mathbf{z}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle = |+\mathbf{x}\rangle \\
\hat{R}(\pi\mathbf{j})|+\mathbf{z}\rangle &= \frac{e^{-i\pi/2}}{\sqrt{2}}|+\mathbf{y}\rangle + \frac{e^{+i\pi/2}}{\sqrt{2}}|-\mathbf{y}\rangle = -\frac{i}{\sqrt{2}}\left[|+\mathbf{y}\rangle - |-\mathbf{y}\rangle\right] = -\frac{i}{\sqrt{2}}\frac{2i}{\sqrt{2}}|-\mathbf{z}\rangle = |-\mathbf{z}\rangle
\end{aligned}$$

(2) This is Problem 3.1 in the textbook.

$$\begin{aligned}
\left[\hat{A}, \hat{B}+\hat{C}\right] &= \hat{A}\hat{B} + \hat{A}\hat{C} - \hat{B}\hat{A} - \hat{C}\hat{A} = \hat{A}\hat{B} - \hat{B}\hat{A} + \hat{A}\hat{C} - \hat{C}\hat{A} = \left[\hat{A}, \hat{B}\right] + \left[\hat{A}, \hat{C}\right] \\
\left[\hat{A}\hat{B}, \hat{C}\right] &= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B} \\
\left[\hat{A}, \hat{B}\hat{C}\right] &= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]
\end{aligned}$$

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Homework Assignment #4

Due at 5pm to the **<u>Grader</u>** on Thursday 11 Feb 2021

(1) In a notation where angular momentum eigenstates are written as usual in the form $|j,m\rangle$, calculate the following matrix elements. Where appropriate, use the Kroneker δ, that is $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$.

a. $\langle j,m|\hat{\mathbf{J}}^2|j,m\rangle$

b. $\langle j',m'|\hat{\mathbf{J}}^2|j,m\rangle$

c. $\langle j',m'|\hat{J}_z|j,m\rangle$

d. $\langle j',m'|\hat{J}_\pm|j,m\rangle$

e. $\langle \frac{1}{2},\frac{1}{2}|J_+|\frac{1}{2},-\frac{1}{2}\rangle$

f. $\langle \frac{3}{2},\frac{1}{2}|J_+|\frac{1}{2},-\frac{1}{2}\rangle$

g. $\langle \frac{3}{2},\frac{1}{2}|J_+|\frac{3}{2},-\frac{1}{2}\rangle$

(2) We show in class on February 9th (and as derived in Section 3.6 of the textbook) that for a normal vector $\mathbf{n} = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi$ in the xy plane, the operator $\hat{S}_n = \hat{\mathbf{S}}\cdot\mathbf{n}$ has the correct eigenvalues $(\pm\hbar/2)$ regardless of the value of ϕ. We also derived eigenstates $|\pm\mathbf{n}\rangle$, given by equations (3.98) and (3.101) in the textbook.

Extend this to three dimensions, using spherical polar coordinates to describe the polarization direction, that is

$$\mathbf{n} = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta$$

Show once again that you get the correct eigenvalues independent of θ and ϕ, and derive the eigenvectors in terms of θ and ϕ. Do this by solving the eigenvalue problem directly, as in Section 3.6.

Solutions HW #4

(1) These are simple, just making use of standard relationships.

a. $\langle j,m|\hat{\mathbf{J}}^2|j,m\rangle = j(j+1)\hbar^2$

b. $\langle j',m'|\hat{\mathbf{J}}^2|j,m\rangle = j(j+1)\hbar^2\delta_{jj'}\delta_{mm'}$

c. $\langle j',m'|\hat{J}_z|j,m\rangle = m\hbar\delta_{jj'}\delta_{mm'}$

d. $\langle j',m'|\hat{J}_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}\hbar\,\delta_{jj'}\delta_{m',m\pm1}$

e. $\langle\frac{1}{2},\frac{1}{2}|J_+|\frac{1}{2},-\frac{1}{2}\rangle = \sqrt{\frac{1}{2}\frac{3}{2}-\left(-\frac{1}{2}\right)\frac{1}{2}}\hbar = \sqrt{2}\hbar$

f. $\langle\frac{3}{2},\frac{1}{2}|J_+|\frac{1}{2},-\frac{1}{2}\rangle = 0$

g. $\langle\frac{3}{2},\frac{1}{2}|J_+|\frac{3}{2},-\frac{1}{2}\rangle = \sqrt{\frac{3}{2}\frac{5}{2}-\left(-\frac{1}{2}\right)\frac{1}{2}}\hbar = 2\hbar$

(2) This is really problem 3.2 in the textbook. The eigenvalue matrix problem is

$$\frac{\hbar}{2}\left[\sigma_x\sin\theta\cos\phi+\sigma_y\sin\theta\sin\phi+\sigma_z\cos\theta\right]\begin{pmatrix}\langle+\mathbf{z}|\mu\rangle\\ \langle-\mathbf{z}|\mu\rangle\end{pmatrix} = \mu\frac{\hbar}{2}\begin{pmatrix}\langle+\mathbf{z}|\mu\rangle\\ \langle-\mathbf{z}|\mu\rangle\end{pmatrix}$$

Inserting the Pauli matrices, this turns into

$$\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{+i\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\langle+\mathbf{z}|\mu\rangle\\ \langle-\mathbf{z}|\mu\rangle\end{pmatrix} = \mu\begin{pmatrix}\langle+\mathbf{z}|\mu\rangle\\ \langle-\mathbf{z}|\mu\rangle\end{pmatrix}$$

so the equation for the eigenvalues becomes

$$(\cos\theta-\mu)(-\cos\theta-\mu)-\sin^2\theta = -(\cos^2\theta-\mu^2)-\sin^2\theta = \mu^2-1=0$$

and the eigenvalues $\mu\hbar/2$ are indeed $\pm\hbar/2$. For $\mu=+1$, we have (with the top equation)

$$\begin{aligned}\cos\theta\langle+\mathbf{z}|+\mathbf{n}\rangle+e^{-i\phi}\sin\theta\langle-\mathbf{z}|+\mathbf{n}\rangle &= \langle+\mathbf{z}|+\mathbf{n}\rangle\\ \frac{\langle-\mathbf{z}|+\mathbf{n}\rangle}{\langle+\mathbf{z}|+\mathbf{n}\rangle} = e^{i\phi}\frac{1-\cos\theta}{\sin\theta} = e^{i\phi}\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} &= e^{i\phi}\frac{\sin(\theta/2)}{\cos(\theta/2)}\end{aligned}$$

From these we can just read off the $\langle\pm\mathbf{z}|+\mathbf{n}\rangle$ because they are already properly normalized. For the $\mu=-1$ eigenvalue, we do the same thing, again with the bottom equation, so

$$\begin{aligned}\cos\theta\langle+\mathbf{z}|-\mathbf{n}\rangle+e^{-i\phi}\sin\theta\langle-\mathbf{z}|-\mathbf{n}\rangle &= -\langle+\mathbf{z}|+\mathbf{n}\rangle\\ \frac{\langle-\mathbf{z}|-\mathbf{n}\rangle}{\langle+\mathbf{z}|-\mathbf{n}\rangle} = -e^{i\phi}\frac{1+\cos\theta}{\sin\theta} = -e^{i\phi}\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} &= -e^{i\phi}\frac{\cos(\theta/2)}{\sin(\theta/2)}\end{aligned}$$

Since $|\pm\mathbf{n}\rangle = |+\mathbf{z}\rangle\langle[+\mathbf{z}|\pm\mathbf{n}\rangle + |-\mathbf{z}\rangle\langle-\mathbf{z}|\pm\mathbf{n}\rangle$ we see that these coefficients agree with the solution given in the problem statement in the textbook.

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Homework Assignment #5

Due at 5pm to the **Grader** on Thursday 18 Feb 2021

(1) *This problem reiterates what we covered in class on Thursday 12 Feb.* Label the combined states of two spin-1/2 particles as we did in class, and also (5.8) in the textbook, as

$$|1\rangle = |+\mathbf{z}, +\mathbf{z}\rangle \qquad |2\rangle = |+\mathbf{z}, -\mathbf{z}\rangle \qquad |3\rangle = |-\mathbf{z}, +\mathbf{z}\rangle \qquad |4\rangle = |-\mathbf{z}, -\mathbf{z}\rangle$$

Then, for the operator $\hat{\mathbf{S}}^2$ from (5.26a) with (5.10),

$$\hat{\mathbf{S}}^2 = \hat{\mathbf{S}}_1^2 + \hat{\mathbf{S}}_2^2 + 2\hat{S}_{1_z}\hat{S}_{2_z} + \hat{S}_{1_+}\hat{S}_{2_-} + \hat{S}_{1_-}\hat{S}_{2_+}$$

calculate the matrix elements $\langle 1|\hat{\mathbf{S}}^2|1\rangle$, $\langle 1|\hat{\mathbf{S}}^2|2\rangle$, $\langle 2|\hat{\mathbf{S}}^2|2\rangle$, $\langle 2|\hat{\mathbf{S}}^2|3\rangle$, $\langle 3|\hat{\mathbf{S}}^2|3\rangle$, and $\langle 4|\hat{\mathbf{S}}^2|4\rangle$. Use this to complete the full 4×4 matrix representation of $\hat{\mathbf{S}}^2$ in the $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ basis. Find the eigenvalues and eigenvectors. Finally, show that the eigenvectors are also eigenvectors of $\hat{S}_z = \hat{S}_{1_z} + \hat{S}_{2_z}$ and derive the eigenvalues.

(2) *I want to try something different on this problem.* The calculation in (1) above is integral to calculating the "hyperfine splitting" in atomic hydrogen. We mentioned this in class, but the result is derived in detail in Section 5.2 of your textbook. The transition between these states corresponds to electromagnetic waves with frequency $\nu = 1420$ MHz, or a wavelength close to 21 cm. It was first measured in the laboratory (not in the Galaxy, as I said mistakenly in class) by Nafe and Nelson, Phys.Rev. **73** (1948) 718. (This seems to be at odds with Footnote 2 on page 146 of your textbook.) Its discovery in the Galaxy was first observed by Ewen and Purcell, Nature **168** (1951) 356.

I want each of you to find a *different* journal article that reports on a measurement that involves the so-called 21 cm line in atomic hydrogen. Most of these will likely be in papers on astrophysics, but I think you can also find some particularly precise laboratory measurements. Any use of this radiation is fine, including papers that show results on galactic rotation curves based on the Doppler shift.

To make sure you each come up with a unique reference, communicate with each other, either using GroupMe or through me or whatever you'd like.

Solutions HW #5

(1) We need to find the matrix elements of all six terms of $\hat{\mathbf{S}}^2$ and add them up. All four states are eigenstates of $\hat{\mathbf{S}}_1^2$ and $\hat{\mathbf{S}}_2^2$ with the same eigenvalue $3\hbar^2/4$, so those are simple. They are also eigenstates of $\hat{S}_{1_z}\hat{S}_{2_z}$ with eigenvalues $+\hbar^2/4$ for $|1\rangle$ and $|4\rangle$, and eigenvalues $-\hbar^2/4$ for $|2\rangle$ and $|3\rangle$. Therefore, the first three terms in $\hat{\mathbf{S}}^2$ contribute only to the diagonal, with $(3/4+3/4+2\times 1/4)\hbar^2 = 2\hbar^2$ for the $1,1$ and $4,4$ elements, and $(3/4+3/4-2\times 1/4)\hbar^2 = \hbar^2$ for the $2,2$ and $3,3$ elements. For the fourth and fifth terms, we need to recall

$$\hat{S}_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar|j,m\pm 1\rangle$$

which gives $\hat{S}_{1_+}\hat{S}_{2_-}|3\rangle = \hbar^2|2\rangle$ and $\hat{S}_{1_-}\hat{S}_{2_+}|2\rangle = \hbar^2|3\rangle$, and zero for each operator on all other states. Therefore, these do not contribute anything to the diagonals, and give $\hbar^2$ for the $2,3$ and $3,2$ matrix elements. The eigenvalue equation $\hat{\mathbf{S}}^2|\mu\rangle = \mu\hbar^2|\mu\rangle$ then becomes, in the $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ matrix representation,

$$\begin{pmatrix} 2-\mu & 0 & 0 & 0 \\ 0 & 1-\mu & 1 & 0 \\ 0 & 1 & 1-\mu & 0 \\ 0 & 0 & 0 & 2-\mu \end{pmatrix}\begin{pmatrix} \langle 1|\mu\rangle \\ \langle 2|\mu\rangle \\ \langle 3|\mu\rangle \\ \langle 4|\mu\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It is easiest to calculate the determinant using "minors", that is, the $1,1$ element times the determinant of the bottom 3×3 matrix, which itself is the $4,4$ element times the determinant of the middle 2×2 matrix, that is

$$(2-\mu)^2[(1-\mu)^2-1](2-\mu)^3\mu = 0$$

so the eigenvalues are $\mu = 2$ three times, and $\mu = 0$. Two of the $\mu = 2$ eigenvectors are obviously $1\rangle$ and $|4\rangle$. The third $\mu = 2$ eigenvector, and the $\mu = 0$ eigenvector (after normalization) are also easy to find, just the sum and difference of the states $|2\rangle$ and $|3\rangle$ divided by $\sqrt{2}$. Noting that $2 = 1(1+1)$ and that $S_z = S_{1_z} + S_{2_z}$ has eigenvalues ± 1 and 0 for the three $\mu = 2$ states, and eigenvalue 0 for the $\mu = 0$ state, we identify the eigenstates as $j = 1$ and $j = 0$ angular momentum eigenstates. That is

$$\begin{aligned} |1,1\rangle &= |1\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}[|2\rangle + |3\rangle] \\ |1,-1\rangle &= |4\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}[|2\rangle - |3\rangle] \end{aligned}$$

(2) If you enter either "1420 MHz" or "21 cm" into Google Scholar, you will see a long list of highly cited articles.

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Homework Assignment #6

Due at 5pm to the <u>**Grader**</u> on Thursday 25 Feb 2021

(1) Work through the details of the discussion in Section 4.4 of the textbook and derive the Rabi formula (4.45) for the probability (as a function of time) of observing the "spin-down" state $|-\mathbf{z}\rangle$, when it is initally in the "spin-up" state $|-\mathbf{z}\rangle$, in terms of the strengths B_0 and B_1 of the static and oscillating magnetic fields, and the oscillation frequency ω, namely

$$|\langle -\mathbf{z}|\psi(t)\rangle|^2 = \frac{\omega_1^2/4}{(\omega_0-\omega)^2+\omega_1^2/4}\sin^2\frac{\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}}{2}t$$

where $\omega_0 = gqB_0/2mc$ and $\omega_1 = gqB_1/2mc$. First, follow the text to obtain the matrix equation (4.41). (Be careful of the signs of the exponents.) Then, explain (as in the text) why it is OK to ignore the terms with $\omega + \omega_0$ and rewrite the simplified matrix equation.

Solve this pair of simplified equations to find $c(t)$ and $d(t)$, but not with assuming that $\omega = \omega_0$. You can do this by assuming that $c(t)$ and $d(t)$ have a $e^{i\alpha t}$ time dependence, and then solve algebraically for α. There are two solutions, and you can use the initial conditions to find the right linear combination. The Rabi equation comes from the inner product of the row matrix (0 1), that is the representation of $\langle -\mathbf{z}|$ in the $|\pm\mathbf{z}\rangle$ basis, and the column matrix of $a(t)$ and $b(t)$, that is the representation of $|\psi(t)\rangle$ in the same basis.

Plot the amplitude of the oscillations as in Figure 4.6 in the textbook, that is the maximum value of the transition probability as a function of the frequency ω of the oscillating field. If you write $\omega = x\omega_0$, then you can plot it against x for values near unity. Make the plot for three different values of the ratio $B_1/B_0 = 10^{-2}$, 10^{-3}, and 10^{-4}. These might be typical ratios for a Magnetic Resonance Imaging (MRI) scanner. Does this help you see MRI machines are valuable medical diagnostic tools?

(2) *This is a general question based on generic states and Hamiltonian. Do not confuse this with "spin" or "angular momentum."* In some three-state system with states labeled $|1\rangle$, $|2\rangle$, and $|3\rangle$, suppose that the matrix representation of the Hamiltonian is

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}$$

a. If the initial state $|\psi(0)\rangle = |2\rangle$, find the state $|\psi(t)\rangle$ as a function of time.

b. If the initial state $|\psi(0)\rangle = |3\rangle$, find the state $|\psi(t)\rangle$ as a function of time.

Express your answers as time-dependent linear combinations of $|1\rangle$, $|2\rangle$, and $|3\rangle$. Remember that it is always easiest to find the time evolution of a state with a time-independent Hamiltonian by expressing the state in terms of the energy eigenstates.

Solutions HW #6

(1) See Problem 4.9 in the textbook. The simplified version of (4.41) gives the two equations

$$i\dot{c}(t) = \frac{\omega_1}{4} e^{i(\omega_0-\omega)t} d(t) \qquad \text{and} \qquad i\dot{d}(t) = \frac{\omega_1}{4} e^{-i(\omega_0-\omega)t} c(t)$$

The first equation gives $d(t) = (4i/\omega_1)e^{-i(\omega_0-\omega)t}\dot{c}(t)$. Use this and the second equation to remove $d(t)$ after taking the time derivative of the first equation, that is

$$\begin{aligned} i\ddot{c}(t) &= i\frac{\omega_1}{4}(\omega_0-\omega)e^{i(\omega_0-\omega)t}\frac{4i}{\omega_1}e^{-i(\omega_0-\omega)t}\dot{c}(t) + \frac{\omega_1}{4}e^{i(\omega_0-\omega)t}\frac{\omega_1}{4i}e^{-i(\omega_0-\omega)t}c(t) \\ \ddot{c}(t) &= i(\omega_0-\omega)\dot{c}(t) - \frac{\omega_1^2}{16}c(t) \end{aligned}$$

Using $c(t) = c_0 e^{i\alpha t}$, find $-\alpha^2 = -\alpha(\omega_0-\omega) - \omega_1^2/16$, or

$$\alpha = \frac{(\omega_0-\omega) \pm \sqrt{(\omega_0-\omega)^2 + \omega_1^2/4}}{2} \equiv \alpha_\pm$$

Therefore, the solution for $c(t)$ takes the form

$$c(t) = c_+ e^{i\alpha_+ t} + c_- e^{i\alpha_- t}$$

and the solution for $d(t)$ becomes

$$d(t) = -\frac{4}{\omega_1}e^{-i(\omega_0-\omega)t}(c_+\alpha_+ e^{i\alpha_+ t} + c_-\alpha_- e^{i\alpha_- t})$$

The initial conditions are $c(0) = 1$ and $d(0) = 0$, since the initial state is $|+\mathbf{z}\rangle$, so

$$c_+ + c_- = 1 \qquad \text{and} \qquad c_+\alpha_+ + c_-\alpha_- = 0$$

Solving these gives

$$c_+ = -\frac{\alpha_-}{\alpha_+ - \alpha_-} = -\frac{\alpha_-}{\sqrt{(\omega_0-\omega)^2 + \omega_1^2/4}} \qquad \text{and} \qquad c_- = \frac{\alpha_+}{\alpha_+ - \alpha_-} = \frac{\alpha_+}{\sqrt{(\omega_0-\omega)^2 + \omega_1^2/4}}$$

I don't see how we are going to enforce $|c|^2 + |d|^2 = 0$, but let's press on. Maybe it's guaranteed from the way we wrote the formalism and the initial conditions. We want to calculate $|\langle -\mathbf{z}|\psi(t)\rangle|^2 = |b(t)|^2 = |d(t)|^2$, so, noting that the $c_\pm$ are real,

$$\begin{aligned} |d(t)|^2 &= \frac{16}{\omega_1^2}\left(c_+\alpha_+ e^{-i\alpha_+ t} + c_-\alpha_- e^{-i\alpha_- t}\right)\left(c_+\alpha_+ e^{i\alpha_+ t} + c_-\alpha_- e^{i\alpha_- t}\right) \\ &= \frac{16}{\omega_1^2}\left(c_+^2\alpha_+^2 + c_-^2\alpha_-^2 + c_+c_-\alpha_+\alpha_-\left[e^{i(\alpha_+-\alpha_-)t} + e^{-i(\alpha_+-\alpha_-)t}\right]\right) \\ &= \frac{16}{\omega_1^2}\left(c_+^2\alpha_+^2 + c_-^2\alpha_-^2 + 2c_+c_-\alpha_+\alpha_-\cos[(\alpha_+-\alpha_-)t]\right) \\ &= \frac{16}{\omega_1^2}\frac{\alpha_+^2\alpha_-^2 + \alpha_-^2\alpha_+^2 + 2\alpha_+^2\alpha_-^2\cos[(\alpha_+-\alpha_-)t]}{(\omega_0-\omega)^2 + \omega_1^2/4} \end{aligned}$$

Now

$$\alpha_+\alpha_- = \frac{(\omega_0-\omega)^2 - (\omega_0-\omega)^2 - \omega_1^2/4}{4} = -\frac{\omega_1^2}{16}$$

Therefore

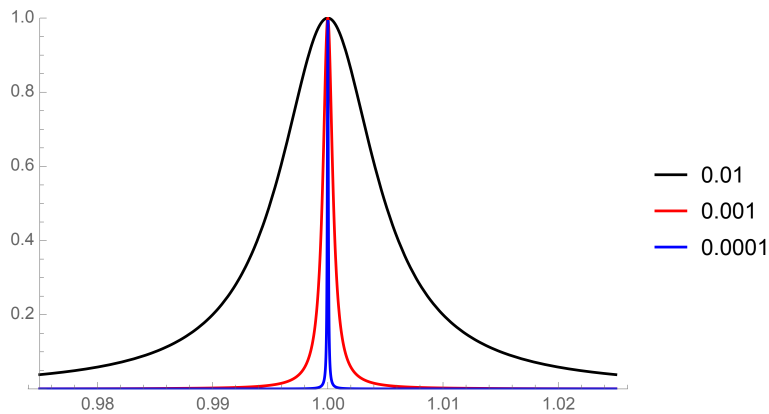
$$\begin{aligned} |d(t)|^2 = |\langle -\mathbf{z}|\psi(t)\rangle|^2 &= \frac{\omega_1^2}{8}\frac{1+\cos[(\alpha_+-\alpha_-)t]}{(\omega_0-\omega)^2+\omega_1^2/4} \\ &= \frac{\omega_1^2}{4}\frac{1}{(\omega_0-\omega)^2+\omega_1^2/4}\sin^2\left(\frac{\alpha_+-\alpha_-}{2}t\right) \\ &= \frac{\omega_1^2/4}{(\omega_0-\omega)^2+\omega_1^2/4}\sin^2\left(\frac{\sqrt{(\omega_0-\omega)^2+\omega_1^2/4}}{2}t\right) \end{aligned}$$

which is Rabi's formula. The "amplitude" is the factor in front of the $\sin^2$ function.

If we write $\omega = x\omega_0$ and $r = \omega_1/\omega_0 = B_1/B_0$, then the amplitude is

$$\frac{\omega_1^2/4}{(\omega_0-\omega)^2+\omega_1^2/4} = \frac{r^2\omega_0^2/4}{(\omega_0-x^2\omega_0)^2+r^2\omega_0^2/4} = \frac{r^2}{4(1-x^2)^2+r^2}$$

Following is a plot of the amplitude versus x made with MATHEMATICA:



(2) *This is really Problem 4.13 in the textbook.* The true lesson here, which I missed when I first started working on the problem, is that it is always easier to find the time dependence by expressing a state in terms of the eigenstates of the Hamiltonian. So, first find the eigenvalues and eigenvectors for the given Hamiltonian matrix. We need to set

$$\begin{vmatrix} E_0 - E & 0 & A \\ 0 & E_1 - E & 0 \\ A & 0 & E_0 - E \end{vmatrix} = (E_0-E)^2(E_1-E)-A^2(E_1-E) = (E_1-E)[(E-E_0)^2-A^2] = 0$$

so the eigenvalues are $E = E_1$, $E = E_0 + A$, and $E = E_0 - A$. Therefore, as represented in the $|1\rangle$, $|2\rangle$, $|3\rangle$ basis, the eigenstates are

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{for} \quad E = E_1 \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix} \quad \text{for} \quad E = E_0 \pm A \equiv E_\pm$$

after solving the matrix equations for the eigenvalues and normalizing the eigenvectors.

For Part (a.), the initial state $|\psi(0)\rangle = |2\rangle = |E_1\rangle$, that is, an energy eigenstate. This is a stationary state with a simple time dependence, namely

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-iE_1t/\hbar}|2\rangle$$

For Part (b.), the initial state $|\psi(0)\rangle = |3\rangle$ is no longer an energy eigenstate, but it is simple to express it as a linear combination of energy eigenstates, namely

$$|\psi(0)\rangle = |3\rangle = \frac{1}{\sqrt{2}}|E_+\rangle - \frac{1}{\sqrt{2}}|E_-\rangle$$

It is now straightforward to find the time dependent state.

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-iE_+t/\hbar}|E_+\rangle - \frac{1}{\sqrt{2}}e^{-iE_-t/\hbar}|E_-\rangle$$

We can also write

$$|E_+\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle \qquad \text{and} \qquad |E_-\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|3\rangle$$

and the time-dependent state becomes

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}e^{-iE_0t/\hbar}e^{-iAt/\hbar}\left[\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle\right] - \frac{1}{\sqrt{2}}e^{-iE_0t/\hbar}e^{+iAt/\hbar}\left[\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|3\rangle\right] \\ &= \frac{1}{2}e^{-iE_0t/\hbar}\left[\left(e^{-iAt/\hbar} - e^{+iAt/\hbar}\right)|1\rangle + \left(e^{-iAt/\hbar} + e^{+iAt/\hbar}\right)|3\rangle\right] \\ &= e^{-iE_0t/\hbar}\cos\left(\frac{A}{\hbar}t\right)|3\rangle - ie^{-iE_0t/\hbar}\sin\left(\frac{A}{\hbar}t\right)|1\rangle \end{aligned}$$

It's good to note that the initial condition is satisfied. I suppose there may be some insight gained by expanding the exponentials and writing everything in terms of sines and cosines, but it seems pretty clear that the state oscillates between $|1\rangle$ and $|3\rangle$.

PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #7

Due at 5pm to the **Grader** on Thursday 4 Mar 2021

(1) Prove that $[\hat{x}^n, \hat{p}_x] = i\hbar n\hat{x}^{n-1}$. You can do this using a relationship you proved on Homework #3, namely $\left[\hat{A}\hat{B}, \hat{C}\right] = \hat{A}\left[\hat{B}, \hat{C}\right] + \left[\hat{A}, \hat{C}\right]\hat{B}$. Then use this result along with the Taylor expansion of $F(x)$ about $x = 0$ to show that

$$[F(\hat{x}), \hat{p}_x] = i\hbar\frac{\partial F}{\partial x}(\hat{x})$$

Finally, show that the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + V(\hat{x}) \qquad \text{gives} \qquad \frac{d\langle p_x\rangle}{dt} = \left\langle -\frac{dV}{dx}\right\rangle$$

and explain how this is equivalent to Newton's Second Law of Motion.

(2) A particle of mass m sits in a one-dimensional infinite well such that $V(x) = 0$ for $0 \le x \le L$. The particle cannot be found outside the well.

a. Find the energy eigenvalues and their eigenfunctions. Compare your answers to the eigenfunctions (6.106) and eigenvalues (6.110) in the textbook.

b. Suppose that the wave function at time $t = 0$ is given by

$$\psi(x, t = 0) = \left(\frac{1+i}{2}\right)\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}$$

for $0 \le x \le L$ and zero elsewhere. Prove that this wave function is properly normalized. You can do this by integrating $\psi^*\psi$ over $0 \le x \le L$, but there is an easier way.

c. Find the wave function $\psi(x, t)$, that is, as a function of time.

d. Calculate the expectation value $\langle E\rangle$ of the energy. You can do this by integrating over $0 \le x \le L$, but there is an easier way. Why is $\langle E\rangle$ independent of time?

e. Calculate the expectation value $\langle x\rangle$ of the particle position, as a function of time. This is a little tedious, unless you us something like MATHEMATICA.

f. What is the probability, as a function of time, that a measurement of the energy will yield the value $\hbar^2\pi^2/2mL^2$. Once again, you can do this by integrating over $0 \le x \le L$, but there is an easier way.

Solutions HW #7

(1) Use the formula given to show a progression, namely

$$\begin{aligned}[\hat{x}^n,\hat{p}_x] &= \hat{x}^{n-1}[\hat{x},\hat{p}_x]+[\hat{x}^{n-1},\hat{p}_x]\hat{x}\\ &= i\hbar\hat{x}^{n-1}+[\hat{x}^{n-1},\hat{p}_x]\hat{x} = i\hbar\hat{x}^{n-1}+\left\{i\hbar\hat{x}^{n-2}+[\hat{x}^{n-2},\hat{p}_x]\hat{x}\right\}\hat{x}\\ &= = i\hbar 2\hat{x}^{n-1}+[\hat{x}^{n-2},\hat{p}_x]\hat{x}^2 = \cdots = i\hbar n\hat{x}^{n-1}\end{aligned}$$

Now writing the function $F(x)$ as a Taylor expansion, we have

$$\begin{aligned}[F(\hat{x}),\hat{p}_x] &= \left[\sum_k \frac{1}{k!}\left.\frac{\partial^k F}{\partial x^k}\right|_{x=0}\hat{x}^k,\hat{p}_x\right] = \sum_k \frac{1}{k!}\left.\frac{\partial^k F}{\partial x^k}\right|_{x=0}\left[\hat{x}^k,\hat{p}_x\right]\\ &= \sum_k \frac{1}{k!}\left.\frac{\partial^k F}{\partial x^k}\right|_{x=0} i\hbar k\hat{x}^{k-1} = i\hbar\frac{\partial F}{\partial x}(\hat{x})\end{aligned}$$

Therefore, the calculation for the rate of change of the momentum expectation value is

$$\frac{d\langle p_x\rangle}{dt} = \frac{i}{\hbar}\left\langle\left[\hat{H},\hat{p}_x\right]\right\rangle = \frac{i}{\hbar}\langle[V(\hat{x}),\hat{p}_x]\rangle = -\left\langle\frac{\partial V}{\partial x}(\hat{x})\right\rangle$$

The negative gradient of the potential energy is called the "force" in classical mechanics. Therefore, this says the force equals $dp/dt = mdV/dt$, i.e. Newton's Second Law of motion.

(2) The differential equation is the same as (6.88) inside the box, so the solution is still (6.100), namely $\psi(x) = A\sin kx + B\cos kx$. However, now we need $\psi(0) = 0$ so $B = 0$, and setting $\psi(L) = A\sin(kL) = 0$ which is the same as (6.104) with $a \to L$. Therefore, the quantization condition is identical (of course), and the energy levels are the same as (6.110), again with $a \to L$. The eigenfunctions are the same as (6.106), but with $x \to x + a/2$ which converts all the cosines to sines, and $n = 1, 2, 3\ldots$. The normalization factor is the same as in (6.108), of course, once more with $a \to L$.

The wave function is a linear combination of the normalized $n = 1$ and $n = 2$ energy eigenfunctions, so it is properly normalized if

$$\left|\frac{1+i}{2}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{(1-i)(1+i)}{4} + \frac{1}{2} = \frac{2}{4} + \frac{1}{2} = 1$$

The integral is done in the MATHEMATICA notebook. Since $e^{-i\hat{H}t/\hbar}|E\rangle = e^{-iEt/\hbar}|E\rangle$, the time dependent wave function is the same as $|\psi(x,0)\rangle$ but with factors $e^{-iE_1 t/\hbar}$ and $e^{-iE_2 t/\hbar}$ in front of the the two terms. The expectation value of the energy must be a constant in time because the Hamiltonian commutes with itself! Explicitly, for $|\psi\rangle = a|E_1\rangle + b|E_2\rangle$,

$$\langle E\rangle = |a|^2E_1 + |b|^2E_2 = \frac{1}{2}\frac{\hbar^2\pi^2}{2mL^2} + \frac{1}{2}\frac{\hbar^2\pi^2}{2mL^2}4 = \frac{5}{2}\frac{\hbar^2\pi^2}{2mL^2}$$

This is also calculated analytically in the MATHEMATICA notebook. For $\langle x\rangle$ we find

$$\langle x\rangle = \frac{8\sqrt{2}L}{9\pi^2}\left[\sin\left(\frac{3\pi^2 t\hbar}{2L^2m}\right) - \cos\left(\frac{3\pi^2 t\hbar}{2L^2m}\right)\right] + \frac{L}{2}$$

Finally, the probability to find the particle in the first energy eigenstate is just $|(1+i)/2|^2 = 1/2$, which is also calculated analytically in the MATHEMATICA notebook.

PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #8

Due at 5pm to the <u>**Grader**</u> on Thursday 11 Mar 2021

(1) The uncertainties in position Δx and momentum Δp_x for some state $|\psi\rangle$ are determined from the expectation values of x, x^2, p_x, and p_x^2 by

$$(\Delta x)^2 = \left\langle (x - \langle x \rangle)^2 \right\rangle = \langle x^2 \rangle - \langle x \rangle^2 \qquad \text{and} \qquad (\Delta p_x)^2 = \left\langle (p_x - \langle p_x \rangle)^2 \right\rangle = \langle p_x^2 \rangle - \langle p_x \rangle^2$$

a. Determine Δx and Δp_x for the ground state $|0\rangle$ of a particle of mass m in a harmonic oscillator potential energy $V(x) = m\omega^2 x^2/2$. Calculate the expectation values by integrating over x. (The normalized ground state wave function is Equation (7.44) in your textbook, and you are welcome to do the integrals using MATHEMATICA, lookup tables, or whatever other modern convenience you have at your disposal.) Form the product $\Delta x \Delta p_x$ and show that it is the minimum value allowed by the uncertainty relation (3.74).

b. Determine Δx and Δp_x for an arbitrary energy eigenstate $|n\rangle$ using operator methods, that is, writing $\hat{x}$ and $\hat{p}_x$ in terms of $\hat{a}$ and $\hat{a}^\dagger$. Show that your answer for $n = 0$ agrees with what you calculated above for the ground state.

(2) Consider a particle of mass m in a harmonic oscillator potential energy $V(x) = m\omega^2 x^2/2$. The initial state is given in terms of energy eigenstates $|n\rangle$ by

$$|\psi(t = 0)\rangle = a|n\rangle + b|n + 1\rangle$$

where a and b are, in principle, complex constants. That is, a superposition of two adjacent energy eigenstates. The state $|\psi(t = 0)\rangle$ is, of course, properly normalized.

a. Calculate the expectation values $\langle x \rangle$ and $\langle p_x \rangle$, both as a function of time. Comment on the result if either a or b is zero, and the effect of the relative phases of a and b on the result. Feel free to modify the notation if you think that is helpful.

b. For $n = 0$ and $a = b = 1/\sqrt{2}$, your answer should follow, and agree with, Example 7.3 in your textbook. Calculate and plot the probability density $|\langle x|\psi(t)\rangle|^2$ for times $t = 0$, $t = \pi/2\omega$, and $t = \pi/\omega$, as shown in Figure 7.10. (If you know how to do it, write a program that animates this.) Comment on how your result compares to $\langle x \rangle$.

Solutions HW #8

(1) For Part (a), use (7.44) $\langle x|0\rangle = (m\omega/\pi\hbar)^{1/4}e^{-m\omega x^2/2\hbar}$ and MATHEMATICA to compute

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} dx\langle 0|x\rangle x\langle x|0\rangle = 0 \\
\langle x^2\rangle &= \int_{-\infty}^{\infty} dx\langle 0|x\rangle x^2\langle x|0\rangle = \frac{\hbar}{2m\omega} \\
\langle p_x\rangle &= \frac{\hbar}{i}\int_{-\infty}^{\infty} dx\langle 0|x\rangle \frac{d}{dx}\langle x|0\rangle = 0 \\
\langle p_x^2\rangle &= -\hbar^2\int_{-\infty}^{\infty} dx\langle 0|x\rangle \frac{d^2}{dx^2}\langle x|0\rangle = \frac{m\omega\hbar}{2}
\end{aligned}$$

Therefore $\Delta x = \sqrt{\hbar/2m\omega}$ and $\Delta p_x = \sqrt{m\omega\hbar/2}$, so $\Delta x\Delta p_x = \hbar/2$. From (3,74), and also (3.71), we expect $\Delta x\Delta p_x \geq |\langle[\hat{x},\hat{p}_x]\rangle|/2 = \hbar/2$, since $[\hat{x},\hat{p}_x] = i\hbar$. Therefore, the ground state of the harmonic oscillator is a minimum uncertainty state.

For Part (b), we use operator methods. *Note that this solution is pretty much done completely in Section 7.5 of the textbook.* Now we use (7.11) and (7.12) to write

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger) \\
\hat{x}^2 &= \frac{\hbar}{2m\omega}(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger 2}) \\
\hat{p}_x &= -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger) \\
\hat{p}_x^2 &= \frac{m\omega\hbar}{2}(-\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}-\hat{a}^{\dagger 2})
\end{aligned}$$

Now since $\hat{a}$ and $\hat{a}^\dagger$ have nonzero matrix elements in the $|n\rangle$ basis only for off diagonal terms, neither they nor $\hat{a}^2$ or $\hat{a}^{\dagger 2}$ will contribute to the matrix elements $\langle x\rangle$, $\langle x^2\rangle$, $\langle p_x\rangle$, and $\langle p_x^2\rangle$. Therefore $\langle x\rangle = 0 = \langle p_x\rangle$. However,

$$\begin{aligned}
& \langle n|\hat{a}\hat{a}^\dagger|n\rangle &&= \sqrt{n+1}\langle n|\hat{a}|n+1\rangle = n+1 \\
\text{and}\quad & \langle n|\hat{a}^\dagger\hat{a}|n\rangle &&= \sqrt{n}\langle n|\hat{a}^\dagger|n-1\rangle = n \\
\text{therefore}\quad & \langle x^2\rangle &&= \frac{\hbar}{2m\omega}(n+1+n) = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) \\
\text{and}\quad & \langle p_x^2\rangle &&= \frac{m\omega\hbar}{2}(n+1+n) = m\omega\hbar\left(n+\frac{1}{2}\right) \\
\text{so}\quad & \Delta x &&= \sqrt{\frac{\hbar}{m\omega}}\sqrt{n+\frac{1}{2}} \\
\text{and}\quad & \Delta p_x &&= \sqrt{m\omega\hbar}\sqrt{n+\frac{1}{2}} \\
\text{giving, finally,}\quad & \Delta x\Delta p_x &&= \left(n+\frac{1}{2}\right)\hbar
\end{aligned}$$

The minimum uncertainty relation is only achieved in the $n = 0$ state.

(2) We follow the procedure we used in class, namely Example 7.3 in the textbook. The time-dependent state is

$$\begin{aligned}|\psi(t)\rangle &= ae^{-iE_nt/\hbar}|n\rangle + be^{-iE_{n+1}t/\hbar}|n+1\rangle \\ &= e^{-ni\omega t/2}\left[a|n\rangle + be^{-i\omega t}|n+1\rangle\right]\end{aligned}$$

Expressions for $\hat{x}$ and $\hat{p}_x$ are given in the first problem. We will need the matrix elements

$$\begin{aligned}\langle\psi(t)|\hat{a}|\psi(t)\rangle &= \left[a^*\langle n| + b^*e^{i\omega t}\langle n+1|\right]\hat{a}\left[a|n\rangle + be^{-i\omega t}|n+1\rangle\right] \\ &= \left[a^*\langle n| + b^*e^{i\omega t/\hbar}\langle n+1|\right]\left[a\sqrt{n}|n-1\rangle + be^{-i\omega t}\sqrt{n+1}|n\rangle\right] \\ &= a^*b\,e^{-i\omega t}\sqrt{n+1} \\ \text{and}\qquad \langle\psi(t)|\hat{a}^\dagger|\psi(t)\rangle &= \left[a^*\langle n| + b^*e^{i\omega t}\langle n+1|\right]\hat{a}^\dagger\left[a|n\rangle + be^{-i\omega t}|n+1\rangle\right] \\ &= \left[a^*\langle n| + b^*e^{i\omega t/\hbar}\langle n+1|\right]\left[a\sqrt{n+1}|n+1\rangle + be^{-i\omega t}\sqrt{n+2}|n+2\rangle\right] \\ &= b^*ae^{i\omega t}\sqrt{n+1}\end{aligned}$$

The expectation values $\langle x\rangle$ and $\langle p\rangle$ follow straightforwardly, namely

$$\begin{aligned}\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(a^*b\,e^{-i\omega t} + b^*a\,e^{i\omega t}\right)\sqrt{n+1} \\ \text{and}\qquad \langle p_x\rangle &= -i\sqrt{\frac{m\omega\hbar}{2}}\left(a^*b\,e^{-i\omega t} - b^*a\,e^{i\omega t}\right)\sqrt{n+1}\end{aligned}$$

Now take advantage of the invitation to "modify the notation." Let's make a real, which involves no loss of generality, just an overall phase factor for $|\psi(t)\rangle$. Then enforce normalization with $b = \sqrt{1-a^2}e^{-i\delta}$, where δ is a real phase. This makes

$$\begin{aligned}a^*b\,e^{-i\omega t} + b^*a\,e^{i\omega t} &= a\sqrt{1-a^2}\left[e^{-i(\omega t+\delta)} + e^{+i(\omega t+\delta)}\right] = 2a\sqrt{1-a^2}\cos(\omega t+\delta) \\ \text{and}\qquad a^*b\,e^{-i\omega t} - b^*a\,e^{i\omega t} &= a\sqrt{1-a^2}\left[e^{-i(\omega t+\delta)} - e^{+i(\omega t+\delta)}\right] = -2ia\sqrt{1-a^2}\sin(\omega t+\delta)\end{aligned}$$

and the expectation values become

$$\begin{aligned}\langle x\rangle &= \sqrt{\frac{2\hbar}{m\omega}}\sqrt{n+1}\,a\sqrt{1-a^2}\,\cos(\omega t+\delta) \\ \text{and}\qquad \langle p_x\rangle &= -\sqrt{2m\omega\hbar}\sqrt{n+1}\,a\sqrt{1-a^2}\,\sin(\omega t+\delta) \\ &= m\frac{d\langle x\rangle}{dt}\end{aligned}$$

as it should be.

For $a = b = 1/\sqrt{2}$, which means $\delta = 0$, and $n = 0$, we get

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t \qquad \text{and} \qquad \langle p_x\rangle = -\sqrt{\frac{m\omega\hbar}{2}}\sin\omega t$$

the first of which agrees with Example 7.3.

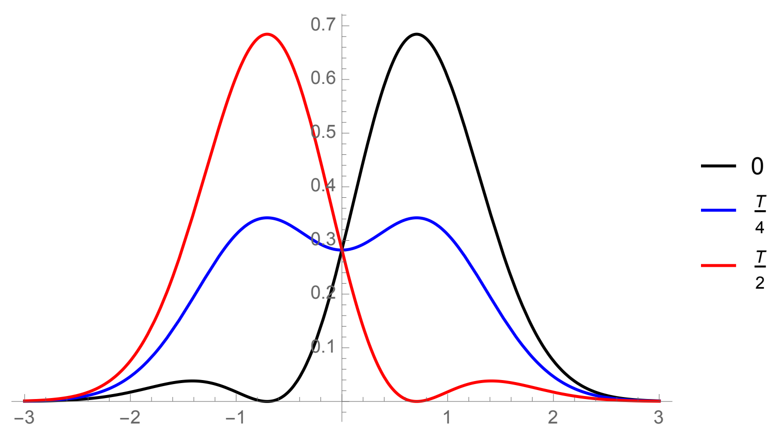
The probability density is

$$
\begin{aligned}
\rho(x,t) &= |\langle x|\psi(t)\rangle|^2 \\
&= |a\langle x|n\rangle + be^{-i\omega t}\langle x|n+1\rangle|^2 \\
&= |a|^2|\langle x|n\rangle|^2 + |b|^2|\langle x|n+1\rangle|^2 + a^*be^{-i\omega t}\langle x|n\rangle^*\langle x|n+1\rangle + ab^*e^{i\omega t}\langle x|n\rangle\langle x|n+1\rangle^* \\
&= \frac{1}{2}|\langle x|0\rangle|^2 + \frac{1}{2}|\langle x|1\rangle|^2 + \langle x|0\rangle\langle x|1\rangle\cos\omega t
\end{aligned}
$$

where we put $n = 0$, $a = b = 1/\sqrt{2}$, and recognize that the wave functions are real. The textbook gives the wave functions for $\langle x|0\rangle$ in (7.44) and $\langle x|1\rangle$ in (7.46), namely

$$
\begin{aligned}
\langle x|0\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \\
\text{and} \qquad \langle x|1\rangle &= \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} xe^{-m\omega x^2/2\hbar}
\end{aligned}
$$

Following is the MATHEMATICA reproduction of Figure 7.10:



PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #9

Due at 5pm to the **Grader** on Thursday 18 Mar 2021

(1) Prove the relation between the square of the orbital angular momentum operator $\hat{\mathbf{L}}^2$ and the 3D position and momentum operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ that we used in class, namely

$$\hat{\mathbf{L}}^2 = \hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = \hat{L}_i \hat{L}_i = \hat{\mathbf{r}}^2 \hat{\mathbf{p}}^2 - (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2 + i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}$$

Make use of the definition $\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$ (Remember: We are using the Einstein summation convention!), the commutation relation $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, and the very useful theorem

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

I'm assigning this problem mostly because I want you to appreciate the power of this notation. It's tricky, though, so here are some tips. First, note that, for two vectors $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{A} \cdot \mathbf{B} = A_i B_i = A_j B_j = \delta_{ij} A_i B_j$ where any two repeated indices are summed, so they are dummy indices. Remember that all three $\hat{x}_i$ commute with each other, as do all $\hat{p}_i$, whereas $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ tells you how to "flip" the order of position and momentum. You also have $\delta_{ij}\delta_{ik} = \delta_{kj}$, and similar examples. Can you convince yourself that $\delta_{kk} = 3$?

So use the expression above for $\varepsilon_{ijk}\varepsilon_{ilm}$, and look at what momenta and positions are "paired" by the Kronecker delta to guide yourself towards the result you want to obtain. You can get to the final answer in eight or ten lines of algebra.

(2) The wave function for a particular particle has the form

$$\psi(\mathbf{r}) = (x + y + z) f(r)$$

where $f(r)$ is some arbitrary function of the radial spherical coordinate. Express $\psi(\mathbf{r})$ in terms of spherical harmonics, tabulated in many places including Equations (9.151), (9.152), and (9.153) in your textbook, and then determine the following:

a. What are the possible results of a measurement of $\hat{\mathbf{L}}^2$, and what are the probabilities for getting these results?

b. What are the possible results of a measurement of $\hat{L}_z$, and what are the probabilities for getting these results?

Solutions HW #9

(1) Note that this is done in MQM3e (3.226) albeit with some notation changes. We have

$$\hat{\mathbf{L}}^2 = \hat{\mathbf{L}}\cdot\hat{\mathbf{L}} = \hat{L}_i\hat{L}_i = (\varepsilon_{ijk}\hat{x}_j\hat{p}_k)(\varepsilon_{ilm}\hat{x}_l\hat{p}_m) = \varepsilon_{ijk}\varepsilon_{ilm}\hat{x}_j\hat{p}_k\hat{x}_l\hat{p}_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\hat{x}_j\hat{p}_k\hat{x}_l\hat{p}_m$$

The first term dots the x's together and the p's together, which is what we want in the first term we are trying to derive, but they are in the wrong order. So flip the inner pair using the commutation relation, and get

$$\delta_{jl}\delta_{km}\hat{x}_j\hat{p}_k\hat{x}_l\hat{p}_m = \delta_{jl}\delta_{km}\hat{x}_j(\hat{x}_l\hat{p}_k - i\hbar\delta_{kl})\hat{p}_m = \hat{x}_j\hat{x}_j\hat{p}_k\hat{p}_k - i\hbar\delta_{jl}\delta_{km}\delta_{kl}\hat{x}_j\hat{p}_m$$

Now $\hat{x}_j\hat{x}_j\hat{p}_k\hat{p}_k = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2$, which is what we were going for. Also $\delta_{jl}\delta_{km}\delta_{kl} = \delta_{jk}\delta_{km} = \delta_{jm}$, so we also have $-i\hbar\delta_{jl}\delta_{km}\delta_{kl}\hat{x}_j\hat{p}_m = -i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}$. Working next on the second term of $\hat{\mathbf{L}}^2$, we notice that it instead dots x's with p's, so focus on the term $(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 = \hat{x}_i\hat{p}_i\hat{x}_j\hat{p}_j$. We have

$$\begin{aligned}
\delta_{jm}\delta_{kl}\hat{x}_j\hat{p}_k\hat{x}_l\hat{p}_m &= \delta_{jm}\delta_{kl}\hat{x}_j\hat{p}_k(\hat{p}_m\hat{x}_l + i\hbar\delta_{lm}) = \delta_{jm}\delta_{kl}\hat{x}_j\hat{p}_k\hat{p}_m\hat{x}_l + i\hbar\delta_{jm}\delta_{kl}\delta_{lm}\hat{x}_j\hat{p}_k \\
&= \delta_{jm}\delta_{kl}\hat{x}_j\hat{p}_m\hat{p}_k\hat{x}_l + i\hbar\delta_{jm}\delta_{kl}\delta_{lm}\hat{x}_j\hat{p}_k \\
&= \delta_{jm}\delta_{kl}\hat{x}_j\hat{p}_m(\hat{x}_l\hat{p}_k - i\hbar\delta_{kl}) + i\hbar\delta_{jm}\delta_{kl}\delta_{lm}\hat{x}_j\hat{p}_k \\
&= \delta_{jm}\delta_{kl}\hat{x}_j\hat{p}_m\hat{x}_l\hat{p}_k - i\hbar\delta_{jm}\delta_{kl}\delta_{kl}\hat{x}_j\hat{p}_m + i\hbar\delta_{jm}\delta_{kl}\delta_{lm}\hat{x}_j\hat{p}_k \\
&= \hat{x}_j\hat{p}_j\hat{x}_k\hat{p}_k - 3i\hbar\hat{x}_j\hat{p}_j + i\hbar\hat{x}_j\hat{p}_j = (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 - 2i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}
\end{aligned}$$

where we recognize that $\delta_{kl}\delta_{kl} = 3$. Finally, putting this all together, we get

$$\hat{\mathbf{L}}^2 = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 - i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - [(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 - 2i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}] = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 - (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 + i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}$$

(2) From (9.152) and the definition of spherical coordinates, we see that

$$\begin{aligned}
Y_{1,0}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \\
Y_{1,1}(\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}(\cos\phi + i\sin\phi)\sin\theta = -\sqrt{\frac{3}{8\pi}}\frac{1}{r}(x+iy) \\
\text{and}\qquad Y_{1,-1}(\theta,\phi) &= \sqrt{\frac{3}{8\pi}}(\cos\phi - i\sin\phi)\sin\theta = \sqrt{\frac{3}{8\pi}}\frac{1}{r}(x-iy) \\
\text{so}\qquad x &= \sqrt{\frac{2\pi}{3}}r\,[Y_{1,-1}(\theta,\phi) - Y_{1,1}(\theta,\phi)] \\
y &= i\sqrt{\frac{2\pi}{3}}r\,[Y_{1,-1}(\theta,\phi) + Y_{1,1}(\theta,\phi)] \\
z &= \sqrt{\frac{4\pi}{3}}rY_{1,0}(\theta,\phi) \\
\text{Therefore}\qquad \langle\mathbf{r}|\psi\rangle &= R(r)N\left[(i-1)\langle\mathbf{r}|1,1\rangle + (i+1)\langle\mathbf{r}|1,-1\rangle + \sqrt{2}\langle\mathbf{r}|1,0\rangle\right] \\
\text{so}\qquad |\psi_L\rangle &= \frac{i-1}{\sqrt{6}}|1,1\rangle + \frac{i+1}{\sqrt{6}}|1,-1\rangle + \sqrt{\frac{1}{3}}|1,0\rangle
\end{aligned}$$

is the (normalized) angular part of the state. A measurement of $\mathbf{L^2}$ can only yield $1(1{+}1)\hbar^2 = 2\hbar^2$ and any measurement of L_z will give $\pm\hbar$ or 0, each with equal 1/3 probabilities.

PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #10

Due at 5pm to the <u>**Grader**</u> on Thursday 25 Mar 2021

(1) The deuteron is the simplest atomic nucleus, the bound state of one proton and one neutron. Experiments show that they are in a relative s-state, that is, $\ell = 0$, and that the binding energy, that is, the energy that needs to be added to separate the proton and neutron, is 2.2 MeV. Experiments also show that the force that binds the proton and neutron can be represented by a finite spherical well of radius $a = 1.7$ fm, that is, twice the RMS radii of the proton and neutron. Use this data to find the depth, in MeV, of the finite spherical well. Some computer programming is required to solve the resulting transcendental equations.

Your textbook outlines a graphical approach to estimate the solution, but I think it is easier to transform the equation to solve for $v \equiv V_0/|E|$ using MATHEMATICA or some other tool. I wrote the appropriately rearranged version of (10.60), plotted both sides to estimate the solution, then used a numerical function to determine the intersection.

When you have the well depth, make a plot that combines the shape of the potential well, a line for the energy level, and the (normalized) wave function $u(r)$. This is Figure 10.9 in your textbook.

You might be amused to know that I solved this problem as an undergraduate in 1976. The numerical solution required me to write a FORTRAN program, typed onto punch cards, which were fed into an IBM 360 Mainframe computer.

(2) An electron is in the ground state of tritium, a nucleus with one proton and two neutrons. Tritium is radioactive, and beta decays to the nucleus ^{3}He, with two protons and one neutron. The decay happens, for all purposes, instantaneously. Calculate the (numerical) probability that the electron remains in the ground state of the $^3\text{He}^+$ ion.

Solutions HW #10

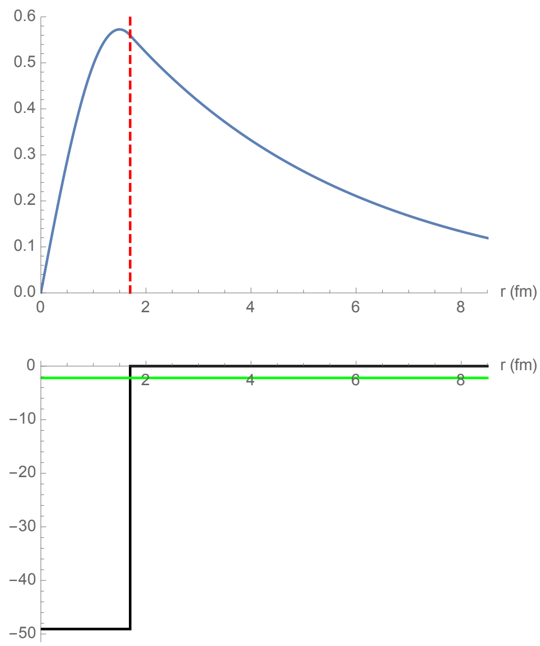
(1) The equations are worked out in Section 10.3 in the textbook. We find V_0 from

$$\tan\left[\sqrt{\frac{2\mu a^2}{\hbar^2}|E|\left(\frac{V_0}{|E|}-1\right)}\right] = -\sqrt{\frac{V_0}{|E|}-1}$$

This lets us solve for the dimensionless quantity $V_0/|E| \equiv v$ because we can just write down the value of $2\mu a^2|E|/\hbar^2$ from the information given. With $\mu = m_p m_n/(m_n+m_p) = m_N/2$,

$$\frac{2\mu a^2}{\hbar^2}|E| = \frac{m_N c^2\, a^2}{\hbar^2 c^2}|E| = \frac{940 \times (1.7)^2}{200^2} \times 2.2 = 0.1494$$

The solution is given in the MATHEMATICA notebook. We find $v = 22.3$ so $V_0 = 49$ MeV. The plot is straightforward. The result from MATHEMATICA is



(2) The probability to measure some eigenstate state $|a\rangle$ when the system is in a state $|\psi\rangle$ is $|\langle a|\psi\rangle|^2$. Denoting one-electron atom energy eigenstates as $|Z; nlm\rangle$, this problem asks what is the probability of finding the $^3\text{He}^+$ ion in the energy eigenstate $|2; 100\rangle$ when it is in the state $|1; 100\rangle$. So, referring to (9.151) and (10.43) we calculate

$$\begin{aligned}
\langle 2; 100|1; 100\rangle &= \int d^3r \langle 2; 100|\mathbf{r}\rangle\langle\mathbf{r}|1; 100\rangle = \int d^3r\, R^*_{2;10}(r) Y^*_{00}(\theta,\phi) R_{1;10}(r) Y_{00}(\theta,\phi) \\
&= \left[\int_0^\infty r^2 dr\, R^*_{2;10}(r) R_{1;10}(r)\right]\left[\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \frac{1}{4\pi}\right] \\
&= \int_0^\infty r^2 dr\, 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0}\, 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} \\
&= \frac{8\sqrt{2}}{a_0^3}\int_0^\infty r^2 dr\, e^{-3r/a_0} = \frac{8\sqrt{2}}{a_0^3} 2! \left(\frac{a_0}{3}\right)^3 = \frac{2^4\sqrt{2}}{3^3}
\end{aligned}$$

Therefore, the probability is $2^9/3^6 = 0.702$. That is, there is a 70.2% chance of finding the $^3\text{He}^+$ ion in its ground state.

PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #11

Due at 5pm to the **<u>Grader</u>** on Thursday 1 Apr 2021

(1) Consider Problem #1 from Homework #10, the assignment from last week. Show that there exists *another* solution that satisfies the same criteria, but with a much deeper well. This is not a physically valid solution only because, in this case, $E = -2.2$ MeV is the energy of the first excited state, and there is no experimental evidence for a more deeply bound ground state in deuterium. Nevertheless, find the energy of this fictional ground state, and make a plot of the wave functions for the ground and first excited states. How do these plots compare to the one you made last week?

(2) Consider a particle of mass m bound by an isotropic harmonic oscillator potential in *two* dimensions, that is

$$V(r) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2(x^2 + y^2)$$

a. Show that the Hamiltonian $\hat{H}$ naturally splits into the sum of two commuting Hamiltonians $\hat{H}_x$ and $\hat{H}_y$, similar to the way we treated the three dimensional case in class.

b. Show that the energy eigenvalues are $E_n = (n + 1)\hbar\omega$ for $n = 0, 1, 2, \ldots$, and express n in terms of the quantum numbers n_x and n_y for $\hat{H}_x$ and $\hat{H}_y$.

c. Discuss the level of energy degeneracy for the lowest few eigenvalues.

d. Define an orbital angular momentum operator in the usual way, namely

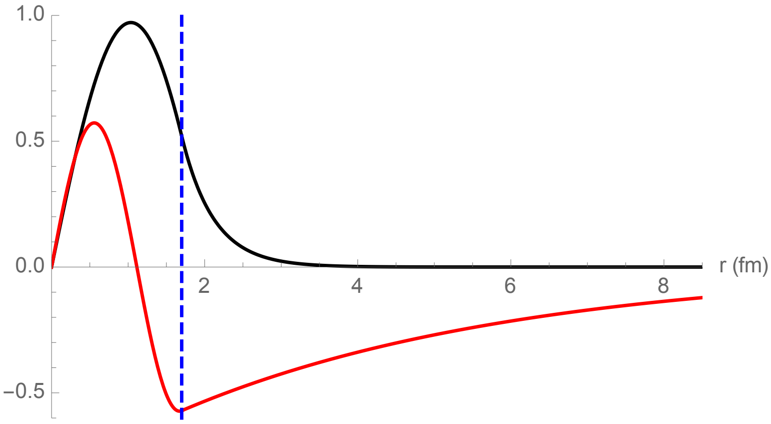
$$\hat{L} \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

You can express $\hat{L}$ in terms of the creation and annihilation operators corresponding to $\hat{H}_x$ and $\hat{H}_y$, as in Equations (7.11) and (7.12) in your textbook. Prove explicitly that $[\hat{H}, \hat{L}] = 0$. What symmetry argument can you give that says this is exactly what you should expect?

e. There are two eigenstates $|n_x, n_y\rangle$ with the same energy $E_1 = 2\hbar\omega$. Determine the correct linear combinations of these that are eigenstates of $\hat{L}$. (They have to exist, because $[\hat{H}, \hat{L}] = 0$, right?) You might be able to guess them, but you can also just diagonalize $\hat{L}$ in the $|n_x, n_y\rangle$ space. What are the eigenvalues of $\hat{L}$?

Solutions HW #11

(1) Repeating the solution from last week, but extending the range of the solution, we find that in addition to $v = V_0/|E| = 22.3$ we also find a solution $v = 154.7$. Using this latter value implies $V_0 = 340.4$ MeV. (See the MATHEMATICA notebook for details.) Solving now for v using this value of V_0, we find the two eigenvalues $E = -242.7$ MeV and, of course, $E = -2.2$ MeV. Here is the plot of the two wave functions:



Just as we expect, the ground state now has a much smaller extent outside the well, and the 2.2 MeV state has a node inside the well.

(2) Clearly $\hat{H} = \hat{H}_x + \hat{H}_y$ where $\hat{H}_x = \hat{p}_x^2/2m + m\omega^2\hat{x}^2/2$ and $\hat{H}_y = \hat{p}_y^2/2m + m\omega^2\hat{y}^2/2$, two operators that commute with each other. Therefore $|E\rangle = |n_x, n_y\rangle$ and $E = E_n = E_{n_x} + E_{n_y}$ with $E_{n_x} = (n_x + 1/2)\hbar\omega$ and $E_{n_y} = (n_y + 1/2)\hbar\omega$ so $E_n = (n + 1)\hbar\omega$ where $n = n_x + n_y$. The $n = 0$ state is nondegenerate, since onl6 $n_x = n_y = 0$ gives $n = 0$. The first excited state ($n = 1$) is twofold degenerate, namely $|1, 0\rangle$ and $|0, 1\rangle$. The three states $|2, 0\rangle$, $|1, 1\rangle$, and $|0, 2\rangle$ all have $n = 2$, so the second excited state has degeneracy 3, and so forth. Now,

$$\begin{aligned}
\hat{L} &= -i\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_x + \hat{a}_x^\dagger)(\hat{a}_y - \hat{a}_y^\dagger) - (\hat{a}_y + \hat{a}_y^\dagger)(\hat{a}_x - \hat{a}_x^\dagger) \\
&= -i\frac{\hbar}{2}(\hat{a}_x\hat{a}_y - \hat{a}_x\hat{a}_y^\dagger + \hat{a}_x^\dagger\hat{a}_y - \hat{a}_x^\dagger\hat{a}_y^\dagger - \hat{a}_y\hat{a}_x + \hat{a}_y\hat{a}_x^\dagger - \hat{a}_y^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_x^\dagger) = i\hbar(\hat{a}_x\hat{a}_y^\dagger - \hat{a}_x^\dagger\hat{a}_y)
\end{aligned}$$

The Hamiltonian is $\hat{H} = \hat{H}_x + \hat{H}_y = (\hat{a}_x^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y + 1)\hbar\omega$, so for $[\hat{H}, \hat{L}] = 0$ we need to show

$$\begin{aligned}
[\hat{a}_x^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y, \hat{a}_x\hat{a}_y^\dagger - \hat{a}_x^\dagger\hat{a}_y] &= [\hat{a}_x^\dagger\hat{a}_x, \hat{a}_x]\hat{a}_y^\dagger - [\hat{a}_x^\dagger\hat{a}_x, \hat{a}_x^\dagger]\hat{a}_y + \hat{a}_x[\hat{a}_y^\dagger\hat{a}_y, \hat{a}_y^\dagger] - \hat{a}_x^\dagger[\hat{a}_y^\dagger\hat{a}_y, \hat{a}_y] \\
&= -\hat{a}_x\hat{a}_y^\dagger - \hat{a}_x^\dagger\hat{a}_y + \hat{a}_x\hat{a}_y^\dagger + \hat{a}_x^\dagger\hat{a}_y = 0
\end{aligned}$$

where we used (7.19) and (7.20). This is expected since $\hat{L}$ generates rotations about the z-axis, and the Hamiltonian is symmetric in the xy plane. Writing $|A\rangle \equiv |1, 0\rangle$ and $|B\rangle \equiv |0, 1\rangle$, we have $\langle A|\hat{L}|A\rangle = i\hbar\langle A|B\rangle = 0 = \langle B|\hat{L}|B\rangle$, but $\langle A|\hat{L}|B\rangle = -i\hbar = -\langle B|\hat{L}|A\rangle$. Therefore, writing the eigenvalues of $\hat{L}$ as $\lambda\hbar$, we find λ from

$$\det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

so the eigenvalues of $\hat{L}$ are $\pm\hbar$. For $+\hbar$, we have $i\langle +\hbar|A\rangle - \langle +\hbar|B\rangle = 0$, while for $-\hbar$ we have $-i\langle -\hbar|A\rangle - \langle -\hbar|B\rangle = 0$, so the normalized eigenvectors are

$$|+\hbar\rangle = \frac{1}{\sqrt{2}}|A\rangle + \frac{i}{\sqrt{2}}|B\rangle \qquad \text{and} \qquad |-\hbar\rangle = \frac{1}{\sqrt{2}}|A\rangle - \frac{i}{\sqrt{2}}|B\rangle$$

PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #12

Due at 5pm to the **Grader** on Thursday 8 Apr 2021

(1) Consider a simple harmonic oscillator in one dimension x for a particle of mass m and with a potential energy function $m\omega^2x^2/2$. Now add a "perturbation" of the form $m\omega_1^2x^2/2$, where $\omega_1 \ll \omega$. Calculate the energy shifts through second order. Of course, this problem can be solved exactly. Expand the exact energy eigenvalues in powers of ω_1/ω to compare to the perturbation expansion.

(2) This problem asks you to calculate the effect of the finite size of the proton on the energy levels of the hydrogen atom. The proton is a "fuzzy" object with a radius close to 1 fm, but for this calculation let's assume it is a uniformly charged sphere of radius R. First, show that, for the radial coordinate $r \leq R$, the electrostatic potential energy is

$$V(r) = -\frac{3}{2}\frac{e^2}{R^3}\left(R^2 - \frac{1}{3}r^2\right)$$

It is probably easiest to do this by using Gauss' Law to get the electric field inside the sphere, integrating to get the potential, and enforcing continuity at $r = R$.

The difference between this potential energy and the point-like form we used to solve the hydrogen atom, can be treated as a perturbation. Calculate the effect of the energy shift on the $1s$, $2s$, and $2p$ states of the hydrogen atom, that is, the states $|100\rangle$, $|200\rangle$, and $|21m\rangle$. Why do you not need to be concerned about the degeneracy in $n = 2$?

It will be best to use MATHEMATICA or some similar application to carry out the necessary integrations. Argue that, physically, $R \ll a_0$ and use this to simply your expressions. It is easy in MATHEMATICA to use the Series function to expand any expression in some small quantity.

If you write your answers for $E^{(1)}_{1,0}$, $E^{(1)}_{2,0}$, and $E^{(1)}_{2,1}$ as a numerical value times e^2/a_0 times a power of R/a_0, you will find that the $l = 0$ and $l = 2$ states have very different dependences on R/a_0. What is the physical reason for this difference?

Solutions HW #12

(1) Let's do the exact solution first. The Hamiltonian is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}m\omega_1^2\hat{x}^2 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m(\omega^2+\omega_1^2)\hat{x}^2$$

Therefore, the energy eigenvalues are

$$E_n = \left(n+\frac{1}{2}\right)\hbar\sqrt{\omega^2+\omega_1^2} = \left(n+\frac{1}{2}\right)\hbar\omega\left(1+\frac{\omega_1^2}{2\omega^2}-\frac{\omega_1^4}{8\omega^4}+\cdots\right)$$

and so we expect perturbation theory to tell us that

$$E_n^{(1)} = \left(n+\frac{1}{2}\right)\hbar\omega\frac{\omega_1^2}{2\omega^2} \qquad \text{and} \qquad E_n^{(2)} = -\left(n+\frac{1}{2}\right)\hbar\omega\frac{\omega_1^4}{8\omega^4}$$

Write the perturbation in terms of creation and annihilation operators, that is

$$\hat{H}_1 = \frac{1}{2}m\omega_1^2\hat{x}^2 = \frac{1}{2}m\omega_1^2\frac{\hbar}{2m\omega}(\hat{a}+\hat{a}^\dagger)^2 = \hbar\omega\frac{\omega_1^2}{4\omega^2}\left(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger^2}\right)$$

We'll need the matrix elements of $\hat{H}_1$ so let's go ahead and calculate

$$\langle k|\left(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+\hat{a}^{\dagger^2}\right)|n\rangle = \sqrt{n(n-1)}\delta_{k,n-2}+(n+1)\delta_{k,n}+n\delta_{k,n}+\sqrt{(n+1)(n+2)}\delta_{k,n+2}$$

The first order energy shift is

$$E_n^{(1)} = \langle n|\hat{H}_1|n\rangle = \hbar\omega\frac{\omega_1^2}{4\omega^2}(2n+1) = \left(n+\frac{1}{2}\right)\hbar\omega\frac{\omega_1^2}{2\omega^2}$$

which agrees with our expectation. The second order energy shift is

$$\begin{aligned}
E_n^{(2)} &= \sum_{k\neq n}\frac{|\langle k|\hat{H}_1|n\rangle|^2}{E_n^{(0)}-E_k^{(0)}} \\
&= (\hbar\omega)^2\frac{\omega_1^4}{16\omega^4}\left[\frac{n(n-1)}{(n+1/2)\hbar\omega-(n-2+1/2)\hbar\omega}+\frac{(n+1)(n+2)}{(n+1/2)\hbar\omega-(n+2+1/2)\hbar\omega}\right] \\
&= \hbar\omega\frac{\omega_1^4}{16\omega^4}\left[\frac{n(n-1)}{2}+\frac{(n+1)(n+2)}{-2}\right] = \hbar\omega\frac{\omega_1^4}{32\omega^4}\left[n^2-n-(n^2+3n+2)\right] \\
&= \hbar\omega\frac{\omega_1^4}{32\omega^4}(-4n-2) = -\left(n+\frac{1}{2}\right)\hbar\omega\frac{\omega_1^4}{8\omega^4}
\end{aligned}$$

which also agrees with our expectation.

(2) The integral form of Gauss' law in Gaussian units is

$$\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi Q_{\text{enc}}$$

Picking a sphere of radius $r \leq R$ for the Gaussian surface, we have

$$\begin{aligned} Q_{\text{enc}} &= \left(\frac{4}{3}\pi r^3 / \frac{4}{3}\pi R^3\right) e = e(r^3/R^3) \\ \text{and} \qquad E(4\pi r^2) &= 4\pi e(r^3/R^3) \qquad \text{so} \qquad E = e\frac{r}{R^3} \end{aligned}$$

The potential *energy* on the electron from the electric potential becomes

$$V(r) = -(-e)\int_0^r E(r')\,dr' + C = \frac{e^2}{2R^3}r^2 + C$$

However, the potential energy needs to be continuous at $r = R$, so

$$\frac{e^2}{2R^3}R^2 + C = -\frac{e^2}{R} \qquad \text{so} \qquad C = -\frac{3e^2}{2R} \qquad \text{and} \qquad V(r) = -\frac{3}{2}\frac{e^2}{R^3}\left(R^2 - \frac{1}{3}r^2\right)$$

In order to write down the correct Hamiltonian $\hat{H}$, we need to subtract away the point-like potential energy function ($\hat{H}_0$) for $r \leq R$ and add in the correct one. Therefore, the perturbation is

$$H_1 = -\frac{3}{2}\frac{e^2}{R^3}\left(R^2 - \frac{1}{3}r^2\right) - \left(-\frac{e^2}{r}\right) = \frac{e^2}{r} - \frac{3}{2}\frac{e^2}{R^3}\left(R^2 - \frac{1}{3}r^2\right) \qquad \text{for } r \leq R$$

The perturbation is zero for $r \geq R$. To get the first order energy corrections to the indicated states, we just take the expectation values of the perturbation. We don't need to be concerned that the $n = 2$ states are degenerate because the perturbation is already diagonal in this subspace. Physically, this is because the perturbation is spherically symmetric, so it commutes with the orbital angular momentum operator.

This is clear from an explicit calculation, which we will do in a moment. You calculate the expectation values in position space, and the integrals over θ and ϕ just give you unity because there are no other angular variables than those in the spherical harmonics. We only need to do the integrals with the radial wave functions $R_{nl}(r)$. These are carried out in the MATHEMATICA notebook. We find, to lowest order in R/a_0,

$$\begin{aligned} E^{(1)}_{1,0} &= \frac{2}{5}\frac{e^2}{a_0}\left(\frac{R}{a_0}\right)^2 \\ E^{(1)}_{2,0} &= \frac{1}{20}\frac{e^2}{a_0}\left(\frac{R}{a_0}\right)^2 \\ E^{(1)}_{2,1} &= \frac{1}{1120}\frac{e^2}{a_0}\left(\frac{R}{a_0}\right)^4 \end{aligned}$$

The expansion in R/a_0 is justified because $R \approx 1$ fm and $a_0 \approx 1$Å, so $R/a_0 \approx 10^{-5}$. Notice that the $l = 0$ states go like $(R/a_0)^2$, while the $l = 1$ state goes like $(R/a_0)^4$. This energy shift is very small, but is *way* smaller for $l = 1$ compared to $l = 0$. This is because the wave function near $r = 0$ goes like r^l. There is only an appreciable piece of the wave function near the origin for $l = 0$ states.

PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #13

Due at 5pm to the **Grader** on Thursday 15 Apr 2021

(1) These questions are meant to associate numbers with atomic hydrogen phenomena.

a. The red $n = 3 \to 2$ Balmer transition has a wavelength $\lambda \approx 656$ nm. Calculate the wavelength difference $\Delta\lambda$ (in nm) between the $3p_{3/2} \to 2s_{1/2}$ and $3p_{1/2} \to 2s_{1/2}$ transitions due to the spin-orbit interaction. Comment on how you might measure this splitting.

b. How large an electric field $\mathcal{E}$ is needed so that the Stark splitting in the $n = 2$ level is the same as the correction from relativistic kinetic energy between the $2s$ and $2p$ levels? How easy or difficult is it to achieve an electric field of this magnitude in the laboratory?

c. The Zeeman effect can be calculated with a "weak" or "strong" magnetic field, depending on the size of the energy shift relative to the spin-orbit splitting. Give examples of a weak and a strong field. How easy or difficult is it to achieve such a magnetic field?

(2) Fill in details and complete the calculation of the "total angular momentum" eigenvalues and eigenstates, in Section 11.5 of your textbook. In particular

a. Find the matrix representation of the operator $2\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ in the basis

$$|1\rangle \equiv |l, m_l, +\mathbf{z}\rangle \qquad \text{and} \qquad |2\rangle \equiv |l, m_l + 1, -\mathbf{z}\rangle$$

b. Show that the eigenvalues of this matrix are $\lambda = l$ and $\lambda = -l - 1$.

c. Show that these correspond to the eigenvalues $j(j+1)\hbar^2$ with $j = l \pm 1/2$ for the operator $\hat{\mathbf{J}}^2$ where $\hat{\mathbf{J}} \equiv \hat{\mathbf{L}} + \hat{\mathbf{S}}$.

d. Show that the expressions (11.93) for the total angular momentum eigenstates

$$|j = l \pm 1/2, m_j\rangle = \sqrt{\frac{l \pm m_j + 1/2}{2l+1}}|l, m_j - 1/2, +\mathbf{z}\rangle \pm \sqrt{\frac{l \mp m_j + 1/2}{2l+1}}|l, m_j + 1/2, -\mathbf{z}\rangle$$

are normalized and give the correct eigenvalues for $\hat{J}_z$.

Solutions HW #13

(1) Part of what goes on here will be getting conscious about the use of CGS units.

(a.) Use (11.96) to get the energy splitting for the $3p$ states.

$$E^{(1)}_{3p_{3/2}} - E^{(1)}_{3p_{1/2}} = \frac{1}{2}mc^2\frac{\alpha^4}{3^3}\frac{1}{3}\left[\frac{1}{2}+1\right] = \frac{1}{2^2 3^3}\frac{1}{137^4}\times 511\ \text{keV} = 1.34\times 10^{-5}\ \text{eV}$$

Since $\lambda = hc/E$ where $E = 13.6\ \text{eV}(1/4 - 1/9) = 1.89$ eV, we have $\Delta\lambda/\lambda = \Delta E/E$. In this case $\Delta E = \Delta^{(1)}_{3p_{3/2}} - \Delta^{(1)}_{3p_{1/2}}$, so $\Delta\lambda = 656\ \text{nm}\times 1.34\times 10^{-5}/1.89 = 656\ \text{nm}\times 7.1\times 10^{-6}$ or $\Delta\lambda = 4.65\times 10^{-3}$ nm. This magnitude splitting is difficult to observe. See for example Fig.6.12 in *Experiments in Modern Physics, 2nd Ed.* by Melissinos and Napolitano. Instrumental effects and thermal broadening give a width $\sim 5\times 10^{-2}$ nm for this line, about ten times too large to see this hyperfine splitting. It is possible to make the measurement with some care, though. See, for example, "Experiments on Hydrogen and Deuterium Fine Structure", by S. Pollack and E. Wong, Am.J.Phys. 39(1971)1386.

(b.) The $2s/2p$ splitting from relativistic kinetic energy is from (11.74):

$$\Delta E_K = \frac{1}{2}mc^2\frac{\alpha^4}{2^3}\left[\frac{1}{1/2} - \frac{1}{3/2}\right] = \frac{1}{12}mc^2\alpha^4 = 1.21\times 10^{-4}\ \text{eV}$$

The Stark splitting from (11.49) is $3ea_0\mathcal{E}$ where $a_0 = 0.53\times 10^{-8}$ cm. This quantity is an energy, so calculate it in SI units. Converting ΔE_K to Joules means multiplying by e, so

$$\mathcal{E} = \frac{1.21\times 10^{-4}}{3\cdot 0.53\times 10^{-10}} = 7.61\times 10^5\ \text{V/m} = 7610\ \text{V/cm}$$

That's a lot of voltage to put across a centimeter, but it can be done. The breakdown voltage in air is about four times higher than this.

(c.) The terms "weak" and "strong" have to do with how large is the Zeeman effect relative to the spin-orbit splitting. As seen in (a.), the magnitude of the spin-orbit splitting is several tens of μeV. From (11.109), the size of the Zeeman effect between different m states is $\sim |e|\hbar B/2mc$, that is, the electron magnetic moment times the magnetic field. Once again, this is an energy, so we can work in SI units. Take the energy scale to be 5×10^{-5} eV= 8×10^{-24} J. The electron magnetic moment is about 9×10^{-24} J/T, so a "weak" magnetic field is much less than 1 T, and a "strong" magnetic field (for the hydrogen $n = 2$ state or so) is much larger than 1 T. Note that fields of several tenths of a Tesla are easy to achieve in the lab, but iron saturates at about 1.5 T so much larger fields are not straightforward to achieve. Nevertheless, we can reach "high" fields for Zeeman splittings in other states where the spin-orbit splitting is smaller. This is in fact known as the Paschen-Back Effect.

(2) From $2\hat{\mathbf{L}}\cdot\hat{\mathbf{S}} = \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + 2L_zS_z$ and $J_\pm|jm\rangle = \sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle$,

$$\begin{aligned}
2\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}|1\rangle &= \sqrt{l(l+1)-m_l(m_l+1)}\hbar^2|l,m_l+1,-\mathbf{z}\rangle + 0 + 2m_l\left(+\frac{1}{2}\right)\hbar^2|l,m_l,+\mathbf{z}\rangle \\
&= \sqrt{l(l+1)-m_l(m_l+1)}\,\hbar^2\,|2\rangle + m_l\,\hbar^2\,|1\rangle \\
2\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}|2\rangle &= 0 + \sqrt{l(l+1)-(m_l+1)m_l}\hbar^2|l,m_l,+\mathbf{z}\rangle + 2(m_l+1)\left(-\frac{1}{2}\right)\hbar^2|l,m_l,+\mathbf{z}\rangle \\
&= \sqrt{l(l+1)-m_l(m_l+1)}\,\hbar^2\,|1\rangle - (m_l+1)|2\rangle
\end{aligned}$$

which gives the matrix (11.86). The characteristic equation is

$$\begin{aligned}
&(m_l-\lambda)(-m_l-l-\lambda) - \sqrt{l(l+1)-m_l(m_l+1)}\sqrt{l(l+1)-m_l(m_l+1)} \\
&= \lambda^2 + \lambda(m_l+1-m_l) - m_l(m_l+1) - l(l+1) + m_l(m_l+1) \\
&= \lambda^2 + \lambda - l(l+1) = 0 \\
\text{so}\qquad \lambda &= \frac{1}{2}\left[-1\pm\sqrt{1+4l(l+1)}\right] = \frac{1}{2}\left[-1\pm(1+2l)\right] = l \quad\text{or}\quad -l-1
\end{aligned}$$

Acting with $\hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2 + \hat{\mathbf{S}}^2 + 2\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}$ on $|lsjm\rangle$ for $\lambda = l$ gives

$$j(j+1) = l(l+1) + \frac{1}{2}\left(\frac{1}{2}+1\right) + l = l^2 + 2l + \frac{3}{4} = \left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right)$$

so $j = l + 1/2$. For $\lambda = -l-1$ we get

$$j(j+1) = l(l+1) + \frac{1}{2}\left(\frac{1}{2}+1\right) - l - 1 = l^2 - \frac{1}{4} = \left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right)$$

and $j = l - 1/2$. It is trivial to show that the given states are normalized since

$$\left(\sqrt{\frac{l\pm m_j+1/2}{2l+1}}\right)^2 + \left(\sqrt{\frac{l\mp m_j+1/2}{2l+1}}\right)^2 = 1$$

Also, since $J_z = L_z + S_z$, and the two components are eigenstates of L_z and S_z, acting on these states gives

$$\begin{aligned}
&J_z|j=l\pm 1/2,m_j\rangle = m_j\hbar|j=l\pm 1/2,m_j\rangle \\
=\;& \sqrt{\frac{l\pm m_j+1/2}{2l+1}}(L_z+S_z)|l,m_j-1/2,+\mathbf{z}\rangle \pm \sqrt{\frac{l\mp m_j+1/2}{2l+1}}(L_z+S_z)|l,m_j+1/2,-\mathbf{z}\rangle \\
=\;& \sqrt{\frac{l\pm m_j+1/2}{2l+1}}(m_j-1/2+1/2)\hbar|l,m_j-1/2,+\mathbf{z}\rangle \\
\pm\;& \sqrt{\frac{l\mp m_j+1/2}{2l+1}}(m_j+1/2-1/2)\hbar|l,m_j+1/2,-\mathbf{z}\rangle \\
=\;& m_j\hbar\left[\sqrt{\frac{l\pm m_j+1/2}{2l+1}}|l,m_j-1/2,+\mathbf{z}\rangle \pm \sqrt{\frac{l\mp m_j+1/2}{2l+1}}|l,m_j+1/2,-\mathbf{z}\rangle\right] \\
=\;& m_j\hbar|j=l\pm 1/2,m_j\rangle
\end{aligned}$$

PHYS3701 Introduction to Quantum Mechanics I Spring 2021

Homework Assignment #14

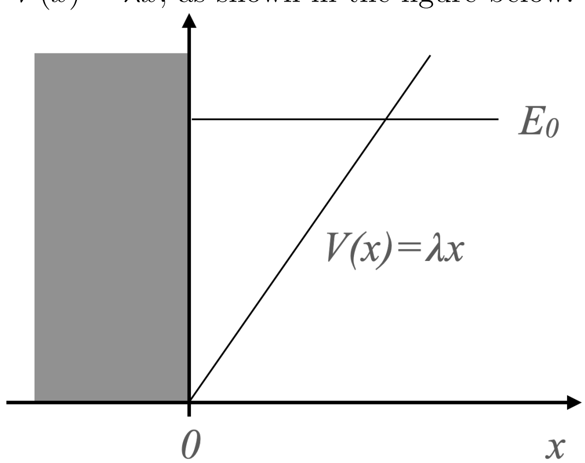
Due at 5pm to the **Grader** on Thursday 22 Apr 2021

(1) The Hamiltonian for a spin-1/2 particle of mass m and charge q in a magnetic field $\mathbf{B}$ is

$$\hat{H} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} = -\frac{gq}{2mc}\hat{\mathbf{S}} \cdot \mathbf{B}$$

where g is the gyromagnetic ratio. The magnetic field has components in the z- and y-directions, and is written as $\mathbf{B} = B_0\hat{\mathbf{k}} + B_1\hat{\mathbf{j}}$. Determine the energy eigenvalues exactly. Then, taking $B_1 \ll B_0$, find the eigenvalues through second order in perturbation theory. Compare the two approaches.

(2) A particle of mass m is bound in one dimension $x \geq 0$ by an infinite wall at $x = 0$ and a linear potential energy $V(x) = \lambda x$, as shown in the figure below.



Use the variational principle to estimate the ground state E_0 using the two different trial functions (i) $\psi(x) = xe^{-x/2a}$ and (ii) $\psi(x) = xe^{-x^2/2a^2}$, where a is a variable parameter. (The calculation can be done analytically, but you are welcome to use Mathematica or some other application.) Why does choice (ii) give you a better approximation to the exact answer? (How do you know it is the better approximation?)

Explain how this can be measured by observing the quantized vertical heights to which a ball can bounce off the floor. Calculate the height to which a neutron must bounce in order to reach the ground state. Compare your answer to the experiment described in "Quantum states of neutrons in the Earth's gravitational field," V.V. Nesvizhevsky, et al., Nature 415 (17 January 2002) 297.

Solutions HW #14

(1) If we write $\mu \equiv gq\hbar/4mc$, then the representation of $\hat{H}$ in the $|\pm\mathbf{z}\rangle$ basis is

$$-\mu(B_0\sigma_z + B_1\sigma_y) = \begin{pmatrix} -\mu B_0 & i\mu B_1 \\ -i\mu B_1 & \mu B_0 \end{pmatrix}$$

so the characteristic equation for the eigenvalues E is

$$(-\mu B_0 - E)(\mu B_0 - E) - \mu^2 B_1^2 = E^2 - \mu^2(B_0^2 - B_1^2) = 0 \qquad \text{i.e.} \qquad E = \pm\mu\sqrt{B_0^2 + B_1^2} \equiv E_\pm$$

We can expand the energy eigenvalues as

$$E_\pm = \pm\mu B_0 \left(1 + \frac{B_1^2}{B_0^2}\right)^{1/2} = \pm\mu B_0 \left(1 + \frac{B_1^2}{2B_0^2}\right) = \pm\mu B_0 \pm \frac{\mu B_1^2}{2B_0}$$

Treating the B_1 term as a perturbation, it is clear that $E_\pm^{(0)} = \pm\mu B_0$ and $|\phi_\pm^{(0)}\rangle = |\pm\mathbf{z}\rangle$, and

$$\hat{H}_1 = -\frac{gqB_1}{2mc} S_y = -\frac{gqB_1}{2mc}\frac{1}{2i}(\hat{S}_+ - \hat{S}_-)$$

Therefore, the first order energy shifts $E_\pm^{(1)} = -(gqB_1/4imc)\langle\pm\mathbf{z}|(\hat{S}_+ - \hat{S}_-)|\pm\mathbf{z}\rangle = 0$, in agreement with the expansion of the exact eigenvalues. For the second order calculation, the sum over "all other states" just includes the one other state. We have

$$\begin{aligned}
E_+^{(2)} &= \frac{|\langle\phi_-^{(0)}|\hat{H}_1|\phi_+^{(0)}\rangle|^2}{E_+^{(0)} - E_-^{(0)}} \\
&= \left(\frac{gqB_1}{4mc}\right)^2 \frac{|\langle-\mathbf{z}|\hat{S}_-|+\mathbf{z}\rangle|^2}{+\mu B_0 - (-\mu B_0)} = \left(\frac{gqB_1}{4mc}\right)^2 \frac{\hbar^2}{2\mu B_0} = \frac{\mu B_1^2}{2B_0} \\
E_-^{(2)} &= \frac{|\langle\phi_+^{(0)}|\hat{H}_1|\phi_-^{(0)}\rangle|^2}{E_-^{(0)} - E_+^{(0)}} \\
&= \left(\frac{gqB_1}{4mc}\right)^2 \frac{|\langle+\mathbf{z}|\hat{S}_+|-\mathbf{z}\rangle|^2}{-\mu B_0 - (+\mu B_0)} = \left(\frac{gqB_1}{4mc}\right)^2 \frac{\hbar^2}{-2\mu B_0} = -\frac{\mu B_1^2}{2B_0}
\end{aligned}$$

which agree with our result from the expansion of the exact eigenvalues.

(2) The approach is straightforward. See the MATHEMATICA notebook for details. You find

$$\text{(i)}\ \ \langle E\rangle = \frac{1}{2a^3}\left(6a^4\lambda + \frac{a\hbar^2}{4m}\right) \qquad \text{(ii)}\ \ \langle E\rangle = \frac{4}{a^3\sqrt{\pi}}\left(\frac{a^4\lambda}{2} + \frac{3a\hbar^2\sqrt{\pi}}{16m}\right)$$

Minimizing with respect to a gives $a_i = 1/(12)^{1/3}(\hbar^2/m\lambda)^{1/3}$ and $a_{ii} = (3\sqrt{\pi}/4)^{1/3}(\hbar^2/m\lambda)^{1/3}$. These give the approximations to the ground state energy as

$$\text{(i)}\ \ \langle E\rangle_{\min} = \left(\frac{3}{2}\right)^{5/3}\left(\frac{\hbar^2\lambda^2}{m}\right)^{1/3} \qquad \text{(ii)}\ \ \langle E\rangle_{\min} = \left(\frac{81}{4\pi}\right)^{1/3}\left(\frac{\hbar^2\lambda^2}{m}\right)^{1/3}$$

The ratio of the second to the first is $(3^4/3^5)^{1/3}(32/4\pi)^{1/3} = 2/(3\pi)^{1/3} = 0.9468$. Since the second choice gives a smaller answer than the first, it must be closer to the exact result. This is to be expected. If the potential function was flat, the wave function would fall as an exponential, i.e. like (i). However, the linear function rises, so we expect the wave function would fall more quickly, and choice (ii) falls more quickly than does (i).

This is like a bouncing ball because the potential energy for a mass m at a height x above the Earth's surface is mgx, that is, $\lambda = mg$. The "floor" is impenetrable, so the region $x \le 0$ is not allowed. For the ground state energy E_0, the height is just $h = E_0/mg$. Using the minimum expectation value from (ii) as our estimate for E_0, we find for a neutron mass

$$h = \left(\frac{81}{4\pi}\right)^{1/3}\left(\frac{\hbar^2}{m^2 g}\right)^{1/3} = 13.8\ \mu\text{m}$$

Following is Figure 4 from the Nesvizhevsky paper. It clearly shows that neutrons cannot pass through their apparatus until they are allowed to bounce as high as 14 μm.