Phys 3701 Intro QM I 2 Feb 2021

Class online today due to Snowstorm

Today: Angular Momentum & Commutation

Reminder: Rotation operator for a state

$$\hat{R}(\phi\hat{k}) = e^{-i\hat{J}_z\phi/\hbar} \quad \text{(about z-axis)}$$

$$\hat{R}(d\phi\,\hat{k}) = \hat{1} - \frac{i}{\hbar}\hat{J}_z\,d\phi \quad \text{Infinitesimal } d\phi$$

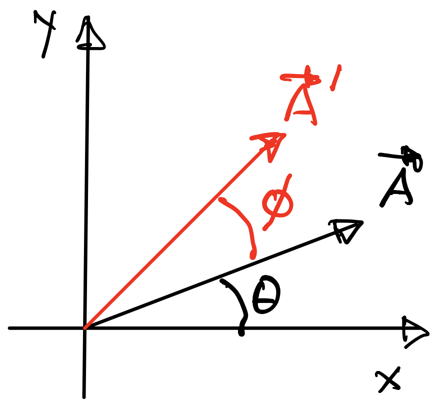
Same thing for rotations about x-axis or y-axis:

$$\hat{R}(d\phi\,\hat{i}) = \hat{1} - \frac{i}{\hbar}\hat{J}_x\,d\phi + O(d\phi^2)$$

$$\hat{R}(d\phi\,\hat{j}) = \hat{1} - \frac{i}{\hbar}\hat{J}_y\,d\phi + O(d\phi^2)$$

$\hat{J}_x, \hat{J}_y, \hat{J}_z$ = "Generators of rotation"
aka "Angular Momentum"

"Rotations Do Not Commute!"



$A_x = A\cos\theta$ $\quad A_y = A\sin\theta$

$A_x' = A\cos(\theta+\phi)$

$A_y' = A\sin(\theta+\phi)$

$$A_x' = A\cos\theta\cos\phi - A\sin\theta\sin\phi$$
$$= A_x\cos\phi - A_y\sin\phi$$

$$A_z' = A_z$$

$$A_y' = \underline{A\sin\theta}\cos\phi + \underline{A\cos\theta}\sin\phi$$
$$= A_x\sin\phi + A_y\cos\phi$$

$$\begin{bmatrix} A_x' \\ A_y' \\ A_z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{S(\phi\hat{k})} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Also

$$S(\phi\hat{i}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$S(\phi\hat{j}) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$S(d\phi\,\hat{i}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{1}{2}d\phi^2 & -d\phi \\ 0 & d\phi & 1-\frac{1}{2}d\phi^2 \end{bmatrix} \quad S(d\phi\,\hat{j}) = \begin{bmatrix} 1-\frac{1}{2}d\phi^2 & 0 & d\phi \\ 0 & 1 & 0 \\ -d\phi & 0 & 1-\frac{1}{2}d\phi^2 \end{bmatrix}$$

$$S(d\phi\,\hat{i})\,S(d\phi\,\hat{j}) = \begin{bmatrix} 1-\frac{1}{2}d\phi^2 & 0 & d\phi \\ d\phi^2 & 1-\frac{1}{2}d\phi^2 & -d\phi + O(d\phi^3) \\ -d\phi & d\phi & 1-d\phi^2 + O(d\phi^4) \end{bmatrix}$$

$$S(d\phi\,\hat{j})\,S(d\phi\,\hat{i}) = \begin{bmatrix} 1-\frac{1}{2}d\phi^2 & d\phi^2 & d\phi \\ 0 & 1-\frac{1}{2}d\phi^2 & -d\phi \\ -d\phi & d\phi & 1-d\phi^2 \end{bmatrix}$$

$$S(d\phi\,\hat{i})\,S(d\phi\,\hat{j}) - S(d\phi\,\hat{j})\,S(d\phi\,\hat{i})$$

$$= \begin{bmatrix} 0 & -d\phi^2 & 0 \\ d\phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\text{But}} \quad S(d\phi^2\,\hat{k}) = \begin{bmatrix} 1 & -d\phi^2 & \\ d\phi^2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{S(d\phi\,\hat{i})\,S(d\phi\,\hat{j}) - S(d\phi\,\hat{j})\,S(d\phi\,\hat{i}) = S(d\phi^2\,\hat{k}) - 1}$$

$$\hat{R}(d\phi\,\hat{i}) = \hat{1} - \frac{i}{\hbar}\hat{J}_x\,d\phi$$

$$\hat{R}(d\phi\,\hat{j}) = \hat{1} - \frac{i}{\hbar}\hat{J}_y\,d\phi$$

$$\hat{R}(d\phi\,\hat{i})\,\hat{R}(d\phi\,\hat{j}) = \hat{1} - \frac{i}{\hbar}\hat{J}_x\,d\phi - \frac{i}{\hbar}\hat{J}_y\,d\phi + \left(\frac{i}{\hbar}\right)^2\hat{J}_x\hat{J}_y\,d\phi^2$$

$$\hat{R}(d\phi\,\hat{j})\,\hat{R}(d\phi\,\hat{i}) = \hat{1} - \frac{i}{\hbar}\hat{J}_y\,d\phi - \frac{i}{\hbar}\hat{J}_x\,d\phi + \left(\frac{i}{\hbar}\right)^2\hat{J}_y\hat{J}_x\,d\phi^2$$

$$\hat{R}(d\phi\,\hat{i})\,\hat{R}(d\phi\,\hat{j}) - \hat{R}(d\phi\,\hat{j})\,\hat{R}(d\phi\,\hat{i})$$

$$= \left(\frac{i}{\hbar}\right)^2\left(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x\right)d\phi^2$$

$$= \hat{R}(d\phi^2\,\hat{k}) - \hat{1}$$

$$= \hat{1} - \frac{i}{\hbar}\hat{J}_z\,d\phi^2 - \hat{1}$$

$$\left(\frac{i}{\hbar}\right)^2\left(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x\right) = -\frac{i}{\hbar}\hat{J}_z = +\frac{i^3}{\hbar}\hat{J}_z$$

$$\boxed{\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x = i\hbar\hat{J}_z}$$

Terminology: $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = -[\hat{B},\hat{A}]$

"Angular Momentum Commutation Relations"

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$$
$$[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$
$$[\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x$$

"cyclic permutation"

$$\vec{J}^2 \equiv J_x^2 + J_y^2 + J_z^2$$

Commutation and Observables

$$\hat{A}\,|a\rangle = a\,|a\rangle$$

$\hat{A}^\dagger = \hat{A}$

Eigenvalue

e.g. $\hat{J}_z |\pm\hat{z}\rangle = \pm\frac{\hbar}{2} |\pm\hat{z}\rangle$

e.g. S_z ment can only give $+\frac{\hbar}{2}, -\frac{\hbar}{2}$

Suppose $[\hat{A}, \hat{B}] = 0 \Rightarrow \hat{A}\hat{B} = \hat{B}\hat{A}$

$$\hat{A}|a\rangle = a|a\rangle \Rightarrow \hat{B}\hat{A}|a\rangle = a\hat{B}|a\rangle$$
$$= \hat{A}\hat{B}|a\rangle$$

"Compatible"

i.e. $\hat{A}\left(\hat{B}|a\rangle\right) = a\left(\hat{B}|a\rangle\right)$ i.e. $\hat{B}|a\rangle = b|a\rangle$