Name: ______________________________

PHYS3701 Intro QM I   S24   Quiz #1   1 Feb 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A certain quantum mechanical system for a spin-1/2 particle is described by the state

$$|\alpha\rangle = a\,|+\hat{\mathbf{z}}\rangle + \frac{i}{\sqrt{3}}\,|-\hat{\mathbf{z}}\rangle$$

where a is a positive real number.

(a) Determine the value of a.

(b) Find the probability that the system would be measured to be in the state $|+\hat{\mathbf{x}}\rangle$.

(c) Calculate what you expect to find for the average $\langle S_z\rangle$ of a large number of measurements of the z-component of the spin vector.

PHYS3701 Intro QM I Quiz #1 Solution

A certain quantum mechanical system for a spin-1/2 particle is described by the state

$$|\alpha\rangle = a\,|+\hat{\mathbf{z}}\rangle + \frac{i}{\sqrt{3}}\,|-\hat{\mathbf{z}}\rangle$$

where a is a positive real number.

(a) Determine the value of a.

(b) Find the probability that the system would be measured to be in the state $|+\hat{\mathbf{x}}\rangle$.

(c) Calculate what you expect to find for the average $\langle S_z\rangle$ of a large number of measurements of the z-component of the spin vector.

The state has to be normalized, so

$$a^2 + \frac{1}{3} = 1 \qquad \text{therefore} \qquad a = \sqrt{\frac{2}{3}}$$

The probability is the square of the projection of the state onto the measured state, so

$$\begin{aligned} |\langle +\hat{\mathbf{x}}|\,\alpha\rangle|^2 &= \left| \left[\frac{1}{\sqrt{2}}\,\langle +\hat{\mathbf{z}}| + \frac{1}{\sqrt{2}}\,\langle -\hat{\mathbf{z}}|\right] \left[\sqrt{\frac{2}{3}}\,|+\hat{\mathbf{z}}\rangle + \frac{i}{\sqrt{3}}\,|-\hat{\mathbf{z}}\rangle\right]\right|^2 \\ &= \left|\frac{1}{\sqrt{3}} + \frac{i}{\sqrt{6}}\right|^2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

The average measured value is just the sum of the different possible values multiplied by the probability for getting that possibility, so

$$\langle S_z\rangle = \left(\frac{\hbar}{2}\right)\left(\frac{2}{3}\right) + \left(-\frac{\hbar}{2}\right)\left(\frac{1}{3}\right) = \frac{\hbar}{6}$$

Name: ______________________________

PHYS3701 Intro QM I S24 Quiz #2 8 Feb 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A spin-1/2 system is governed by the Hamiltonian $H = \omega S_z$. If the system starts out in the state $|+\hat{\mathbf{y}}\rangle$, find the probability at some time t that a measurement finds the system in the state $|-\hat{\mathbf{z}}\rangle$. Explain why your answer does (or does not) make sense.

A spin-1/2 system is governed by the Hamiltonian $H = \omega S_z$. If the system starts out in the state $|+\hat{\mathbf{y}}\rangle$, find the probability at some time t that a measurement finds the system in the state $|-\hat{\mathbf{z}}\rangle$. Explain why your answer does (or does not) make sense.

The answer is a very straightforward application of the time evolution operator. The time evolved state is

$$\begin{aligned} |\alpha;t\rangle &= e^{-iHt/\hbar}|\alpha\rangle = e^{-i\omega S_z t/\hbar}\,|+\hat{\mathbf{y}}\rangle \\ &= e^{-i\omega S_z t/\hbar}\left[\frac{1}{\sqrt{2}}\,|+\hat{\mathbf{z}}\rangle + \frac{i}{\sqrt{2}}\,|-\hat{\mathbf{z}}\rangle\right] = \frac{e^{-i\omega t/2}}{\sqrt{2}}\,|+\hat{\mathbf{z}}\rangle + i\frac{e^{i\omega t/2}}{\sqrt{2}}\,|-\hat{\mathbf{z}}\rangle \end{aligned}$$

The probability to find the system in the $|-\hat{\mathbf{z}}\rangle$ state is

$$\left|i\frac{e^{i\omega t/2}}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

which is what you expect. Starting in the $|+\hat{\mathbf{y}}\rangle$ state, with a field in the z-direction, the spin precesses around the z-axis, with a changing linear combination of $|+\hat{\mathbf{y}}\rangle$ and $|-\hat{\mathbf{y}}\rangle$, which is the same as some other linear combination of $|+\hat{\mathbf{x}}\rangle$ and $|-\hat{\mathbf{x}}\rangle$. For any of these, the probability of measuring the spin in the z-direction is 1/2.

Name: ______________________________

PHYS3701 Intro QM I   S24   Quiz #3   15 Feb 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Prove the following, where p is the canonically conjugate momentum to x:

$$[x, F(p)] = i\hbar \frac{dF}{dp}$$

You can do this by first proving that $[x, p^n] = i\hbar n p^{n-1}$ by showing it is true for $n = 1$ and then showing that if it is true for n, then it is true for $n + 1$, and then writing $F(p)$ as a Taylor series, that is $F(p) = \sum_n a_n p^n$ where the a_n are just some constants.

PHYS3701 Intro QM I Quiz #3 Solution

Prove the following, where p is the canonically conjugate momentum to x:

$$[x, F(p)] = i\hbar \frac{dF}{dp}$$

You can do this by first proving that $[x, p^n] = i\hbar n p^{n-1}$ by showing it is true for $n = 1$ and then showing that if it is true for n, then it is true for $n + 1$, and then writing $F(p)$ as a Taylor series, that is $F(p) = \sum_n a_n p^n$ where the a_n are just some constants.

Do this by following the same procedure as for Homework #4 Problem (3).

Proving that $[x, p^n] = i\hbar n p^{n-1}$ for $n = 1$ is just stating the canonical commutation relation, namely $[x, p] = i\hbar$. Writing it down for $n + 1$ and using the relation for n gives

$$\begin{aligned}
[x, p^{n+1}] &= xp^{n+1} - p^{n+1}x \\
&= (xp)p^n - p^{n+1}x \\
&= (i\hbar + px)p^n - p^{n+1}x \\
&= i\hbar p^n + pxp^n - p^{n+1}x \\
&= i\hbar p^n + p(xp^n - p^n x) \\
&= i\hbar p^n + p[x, p^n] \\
&= i\hbar p^n + p(i\hbar n p^{n-1}) \\
&= i\hbar (n+1) p^n \qquad \text{QED}
\end{aligned}$$

Now it is straightforward to get the final result.

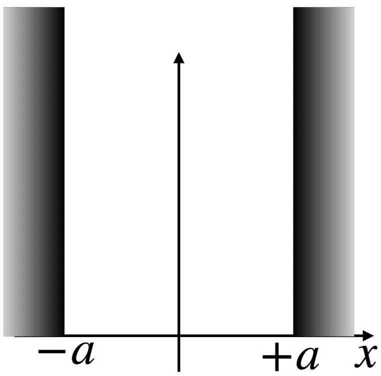
$$[x, F(p)] = \left[x, \sum_n a_n p^n\right] = \sum_n a_n \, [x, p^n] = i\hbar \sum_n n a_n p^{n-1} = i\hbar \frac{dF}{dp}$$

Name: ____________________

PHYS3701 Intro QM I   S24   Quiz #4   22 Feb 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

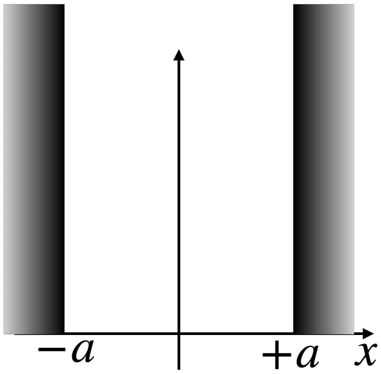


A particle of mass m is confined to an infinite square potential well in one dimension and that extends over $-a \leq x \leq a$. Writing the wave function as

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

show that the boundary conditions imply the energy levels are the same as we derived in class for an infinite square well extending over $0 \leq x \leq L$, and that $\psi(-x) = \pm\psi(x)$. *Hint: Recall how matrices solve simultaneous algebraic equations.*

PHYS3701 Intro QM I Quiz #4 <u>Solution</u>



A particle of mass m is confined to an infinite square potential well in one dimension and that extends over $-a \le x \le a$. Writing the wave function as

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

show that the boundary conditions imply the energy levels are the same as we derived in class for an infinite square well extending over $0 \le x \le L$, and that $\psi(-x) = \pm\psi(x)$. *Hint: Recall how matrices solve simultaneous algebraic equations.*

The wave function has to vanish at $x = \pm a$, so

$$\begin{aligned} Ae^{ika} + Be^{-ika} &= 0 \\ Ae^{-ika} + Be^{ika} &= 0 \end{aligned}$$

Written as a matrix equation for A and B, this means that we must have

$$\begin{vmatrix} e^{ika} & e^{-ika} \\ e^{-ika} & e^{ika} \end{vmatrix} = e^{2ika} - e^{-2ika} = 2i\sin(2ka) = 0$$

This implies that $2ka = n\pi$ for $n = 1, 2, 3\ldots$, so the energy levels are

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 (ka)^2}{2ma^2} = \frac{\hbar^2\pi^2}{8ma^2}n^2 = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

where $L = 2a$, and this is the result we got in class.

Now $2ka = n\pi$ means $ka = (m+1/2)\pi$, $m = 1, 2, 3\ldots$ if n is odd, or $ka = m\pi$, $m = 1, 2, 3\ldots$ if n is even. For the case of n odd, we have

$$Ae^{im\pi}e^{i\pi/2} + Be^{-im\pi}e^{-i\pi/2} = i\cos(m\pi)[A - B] = 0$$

so $B = A$ in which case $\psi(x) = Ae^{ikx} + Ae^{-ikx}$ and $\psi(-x) = \psi(x)$. For the case of n even,

$$Ae^{im\pi} + Be^{-im\pi} = \cos(m\pi)[A + B] = 0$$

so $B = -A$ in which case $\psi(x) = Ae^{ikx} - Ae^{-ikx}$ and $\psi(-x) = -\psi(x)$.

Name: ______________________________

PHYS3701 Intro QM I S24 Quiz #5 29 Feb 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the expectation value of the kinetic energy $p^2/2m$ in an eigenstate $|n\rangle$ of the simple harmonic oscillator in one dimension. Express your answer as a factor times the energy eigenvalue E_n. Use your result to infer (or otherwise calculate) the expectation value of the potential energy $m\omega^2 x^2/2$. You are welcome to use any results from your homework that you would find useful, but refer to the relevant problem solution.

PHYS3701 Intro QM I Quiz #5 Solution

Find the expectation value of the kinetic energy $p^2/2m$ in an eigenstate $|n\rangle$ of the simple harmonic oscillator in one dimension. Express your answer as a factor times the energy eigenvalue E_n. Use your result to infer (or otherwise calculate) the expectation value of the potential energy $m\omega^2 x^2/2$. You are welcome to use any results from your homework that you would find useful, but refer to the relevant problem solution.

We are asked to evaluate $\langle n|p^2/2m|n\rangle = \langle n|p^2|n\rangle/2m$. However, in HW#6 Problem 5, we calculated

$$\langle n|p^2|n\rangle = -\frac{m\hbar\omega}{2}\langle n|(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger)|n\rangle = -\frac{m\hbar\omega}{2}[-(n+1) - n] = m\hbar\omega\left(n + \frac{1}{2}\right)$$

We also know, as we derived in class

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Therefore the expectation value of the kinetic energy is just

$$\frac{1}{2m}\langle p^2\rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) = \frac{1}{2}E_n$$

This means that the expectation value of the potential energy must be the rest, namely also $E_n/2$. More formally

$$\langle H\rangle = E_n = \frac{1}{2m}\langle p^2\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle \qquad \text{so} \qquad \frac{1}{2}m\omega^2\langle x^2\rangle = E_n - \frac{1}{2}E_n = \frac{1}{2}E_n$$

You could also derive this result, again using Problem 5 in HW#6, that is

$$\frac{1}{2}m\omega^2\langle n|x^2|n\rangle = \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}\langle n|(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)|n\rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) = \frac{1}{2}E_n$$

Name: ______________________________

PHYS3701 Intro QM I S24 Quiz #6 14 Mar 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Calculate the result of rotating the state $|{+\hat{\mathbf{x}}}\rangle$ by 90° about the z-axis, that is $\mathcal{D}(\hat{z}, \pi/2)\,|{+\hat{\mathbf{x}}}\rangle$, and show that you get the answer you expect. You are welcome to work this out in either operator language or using the matrix representation in the $|{\pm\hat{\mathbf{z}}}\rangle$ basis.

PHYS3701 Intro QM I Quiz #6 Solution

Calculate the result of rotating the state $|+\hat{\mathbf{x}}\rangle$ by 90° about the z-axis, that is $\mathcal{D}(\hat{z},\pi/2)\,|+\hat{\mathbf{x}}\rangle$, and show that you get the answer you expect. You are welcome to work this out in either operator language or using the matrix representation in the $|\pm\hat{\mathbf{z}}\rangle$ basis.

You expect, of course, to get $|+\hat{\mathbf{y}}\rangle$, perhaps with a phase factor in front. In operator language

$$\begin{aligned}\mathcal{D}(\hat{z},\pi/2)\,|+\hat{\mathbf{x}}\rangle &= e^{-i\pi S_z/2\hbar}\frac{1}{\sqrt{2}}\left(|+\hat{\mathbf{z}}\rangle + |-\hat{\mathbf{z}}\rangle\right) = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}\,|+\hat{\mathbf{z}}\rangle + e^{+i\pi/4}\,|-\hat{\mathbf{z}}\rangle\right)\\ &= e^{-i\pi/4}\frac{1}{\sqrt{2}}\left(|+\hat{\mathbf{z}}\rangle + e^{+i\pi/2}\,|-\hat{\mathbf{z}}\rangle\right) = e^{-i\pi/4}\frac{1}{\sqrt{2}}\left(|+\hat{\mathbf{z}}\rangle + i\,|-\hat{\mathbf{z}}\rangle\right) = e^{-i\pi/4}\,|+\hat{\mathbf{y}}\rangle\end{aligned}$$

In the $|\pm\hat{\mathbf{z}}\rangle$ representation, the rotation operator becomes

$$\underline{\underline{\mathcal{D}}}(\hat{z},\pi/2) = \underline{\underline{1}}\cos\frac{\pi}{4} - i\underline{\underline{\sigma}}_z\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}$$

so $\mathcal{D}(\hat{z},\pi/2)\,|+\hat{\mathbf{x}}\rangle$ becomes

$$\begin{aligned}\frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 1-i \\ 1+i \end{bmatrix} = \frac{1-i}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ (1+i)/(1-i) \end{bmatrix}\\ &= \frac{1-i}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ (1+i)^2/2 \end{bmatrix}\\ &= \frac{1-i}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-i\pi/4}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}\end{aligned}$$

which is the same result as above.

Name: ______________________

PHYS3701 Intro QM I S24 Quiz #7 21 Mar 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Two spin-1/2 particles are emitted from a spin-zero state. What is the probability that an observer measures particle #1 to be in the $|+\hat{\mathbf{z}}\rangle$ state, while particle #2 is observed to be in the $|+\hat{\mathbf{x}}\rangle$ state? *You are welcome to say what the answer is without a calculation, but it has to be accompanied by a valid explanation. Doing the calculation is perfectly fine, of course.*

PHYS3701 Intro QM I Quiz #7 <u>Solution</u>

Two spin-1/2 particles are emitted from a spin-zero state. What is the probability that an observer measures particle #1 to be in the $|+\hat{\mathbf{z}}\rangle$ state, while particle #2 is observed to be in the $|+\hat{\mathbf{x}}\rangle$ state? *You are welcome to say what the answer is without a calculation, but it has to be accompanied by a valid explanation. Doing the calculation is perfectly fine, of course.*

The answer has to be 1/4 because we know particle #2 is in the $|-\hat{\mathbf{z}}\rangle$ state, which is equal parts $|+\hat{\mathbf{x}}\rangle$ and $|-\hat{\mathbf{x}}\rangle$, and the probability of measuring particle #1 to be in the $|+\hat{\mathbf{z}}\rangle$ state is 1/2. However, it's easy enough to work out the calculation. Express the initial state in the $|\pm\hat{\mathbf{x}}\rangle$ basis because by now, we know by heart how to write $|\pm\hat{\mathbf{x}}\rangle$ in terms of $|\pm\hat{\mathbf{z}}\rangle$. The two-particle state is

$$\begin{aligned}
|\alpha\rangle &= \frac{1}{\sqrt{2}}\left[|+\hat{\mathbf{x}}\rangle \otimes |-\hat{\mathbf{x}}\rangle - |-\hat{\mathbf{x}}\rangle \otimes |+\hat{\mathbf{x}}\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|+\hat{\mathbf{z}}\rangle + |-\hat{\mathbf{z}}\rangle\right) \otimes |-\hat{\mathbf{x}}\rangle - \frac{1}{\sqrt{2}}\left(|+\hat{\mathbf{z}}\rangle - |-\hat{\mathbf{z}}\rangle\right) \otimes |+\hat{\mathbf{x}}\rangle\right] \\
&= \frac{1}{2}\left[|+\hat{\mathbf{z}}\rangle \otimes |-\hat{\mathbf{x}}\rangle + |-\hat{\mathbf{z}}\rangle \otimes |-\hat{\mathbf{x}}\rangle - |+\hat{\mathbf{z}}\rangle \otimes |+\hat{\mathbf{x}}\rangle + |-\hat{\mathbf{z}}\rangle \otimes |+\hat{\mathbf{x}}\rangle\right]
\end{aligned}$$

Therefore the probability of observing the state $|+\hat{\mathbf{z}}\rangle \otimes |+\hat{\mathbf{x}}\rangle$ is

$$\left|\left(\langle+\hat{\mathbf{z}}| \otimes \langle+\hat{\mathbf{x}}|\right)|\alpha\rangle\right|^2 = \left|\frac{1}{2}[0+0-1+0]\right|^2 = \frac{1}{4}$$

Name: ____________________

PHYS3701 Intro QM I S24 Quiz #8 28 Mar 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Two spin-1/2 particles are emitted from the state $|j, m\rangle = |1, 1\rangle$ in the z-basis, that is a linear combination of the four states $|\pm\hat{\mathbf{z}}\rangle \otimes |\pm\hat{\mathbf{z}}\rangle$. What is the probability that an observer measures particle #1 to be in the $|+\hat{\mathbf{x}}\rangle$ state, while particle #2 is observed to be in the $|-\hat{\mathbf{x}}\rangle$ state? Don't just give the answer; show how to calculate it.

PHYS3701 Intro QM I Quiz #8 Solution

Two spin-1/2 particles are emitted from the state $|j,m\rangle = |1,1\rangle$ in the z-basis, that is a linear combination of the four states $|\pm\hat{\mathbf{z}}\rangle \otimes |\pm\hat{\mathbf{z}}\rangle$. What is the probability that an observer measures particle #1 to be in the $|+\hat{\mathbf{x}}\rangle$ state, while particle #2 is observed to be in the $|-\hat{\mathbf{x}}\rangle$ state? Don't just give the answer; show how to calculate it.

As we showed in class, $|1,1\rangle = |+\hat{\mathbf{z}}\rangle \otimes |+\hat{\mathbf{z}}\rangle$. The observed state is

$$
\begin{aligned}
|+\hat{\mathbf{x}}\rangle \otimes |-\hat{\mathbf{x}}\rangle &= \left(\frac{1}{\sqrt{2}}|+\hat{\mathbf{z}}\rangle + \frac{1}{\sqrt{2}}|-\hat{\mathbf{z}}\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|+\hat{\mathbf{z}}\rangle - \frac{1}{\sqrt{2}}|-\hat{\mathbf{z}}\rangle\right) \\
&= \frac{1}{2}\left(|+\hat{\mathbf{z}}\rangle \otimes |+\hat{\mathbf{z}}\rangle - |+\hat{\mathbf{z}}\rangle \otimes |-\hat{\mathbf{z}}\rangle + |-\hat{\mathbf{z}}\rangle \otimes |+\hat{\mathbf{z}}\rangle - |-\hat{\mathbf{z}}\rangle \otimes |-\hat{\mathbf{z}}\rangle\right)
\end{aligned}
$$

Therefore the probability of observing $|+\hat{\mathbf{x}}\rangle \otimes |-\hat{\mathbf{x}}\rangle$ is

$$
\left|\left(\langle+\hat{\mathbf{x}}| \otimes \langle-\hat{\mathbf{x}}|\right)\left(|+\hat{\mathbf{z}}\rangle \otimes |+\hat{\mathbf{z}}\rangle\right)\right|^2 = \left|\frac{1}{2}(1-0+0-0)\right|^2 = \frac{1}{4}
$$

Name: ____________________

PHYS3701 Intro QM I S24 Quiz #9 11 Apr 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Prove that $\vec{L}\times\vec{L} = i\hbar\vec{L}$ where $\vec{L}$ is the orbital angular momentum operator. You will get full credit if you prove this is true component by component. For 50% extra credit, prove it using (with the usual summation convention) $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$ and writing the components of $\vec{L}$ as $L_i = \epsilon_{ijk}r_jp_k$, of course being careful to choose summation indices that do not clash.

Prove that $\vec{L}\times\vec{L} = i\hbar\vec{L}$ where $\vec{L}$ is the orbital angular momentum operator. You will get full credit if you prove this is true component by component. For 50% extra credit, prove it using (with the usual summation convention) $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$ and writing the components of $\vec{L}$ as $L_i = \epsilon_{ijk}r_jp_k$, of course being careful to choose summation indices that do not clash.

This is very simple if we just look at it by individual components. For example, with the z-component

$$\left(\vec{L}\times\vec{L}\right)_z = L_xL_y - L_yL_x = [L_x, L_y] = i\hbar L_z$$

and similarly for the x- and y-components.

It is not hard to do this with the summation convention for all three components at once.

$$\begin{aligned}
\left(\vec{L}\times\vec{L}\right)_i &= \epsilon_{ijk}L_jL_k \\
&= \epsilon_{ijk}\epsilon_{jmn}r_mp_n\epsilon_{klq}r_lp_q \\
&= \epsilon_{kij}\epsilon_{klq}\epsilon_{jmn}r_mp_nr_lp_q \\
&= (\delta_{il}\delta_{jq} - \delta_{iq}\delta_{jl})\epsilon_{jmn}r_mp_nr_lp_q \\
&= \epsilon_{jmn}r_mp_nr_ip_j - \epsilon_{jmn}r_mp_nr_jp_i \\
&= \epsilon_{jmn}r_mp_n(r_ip_j - r_jp_i) \\
&= \epsilon_{jmn}r_mp_n[r_i, p_j] \\
&= \epsilon_{jmn}r_mp_n\, i\hbar\delta_{ij} \\
&= i\hbar\epsilon_{imn}r_mp_n \\
&= i\hbar L_i
\end{aligned}$$

Name: ______________________________

PHYS3701 Intro QM I S24 Quiz #10 18 Apr 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A system in three dimensional space has a wave function $\psi(\vec{r}) = N f(r)\, z^2$ where r and z are the usual spatial coordinates, $f(r)$ is an arbitrary function, and N is a constant.

(a) Show that the state is a mixture of angular momentum eigenstates $|lm\rangle$ for $l = 0$ and $l = 2$. You should use some table of spherical harmonics from your favorite source.

(b) Find the probabilities that a measurement observes each possible eigenstate $|lm\rangle$.

PHYS3701 Intro QM I Quiz #10 Solution

A system in three dimensional space has a wave function $\psi(\vec{r}) = Nf(r)\,z^2$ where r and z are the usual spatial coordinates, $f(r)$ is an arbitrary function, and N is a constant.

(a) Show that the state is a mixture of angular momentum eigenstates $|lm\rangle$ for $l=0$ and $l=2$. You should use some table of spherical harmonics from your favorite source.

(b) Find the probabilities that a measurement observes each possible eigenstate $|lm\rangle$.

The factor z^2 suggests we look at the spherical harmonics with $l=2$. In fact

$$Y_2^0(\theta,\phi) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}(3\cos^2\theta - 1) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}\frac{3z^2-r^2}{r^2}$$

Noting that $Y_0^0(\theta,\phi) = 1/\sqrt{4\pi}$, we write

$$Y_2^0(\theta,\phi) + \frac{\sqrt{5}}{2}Y_0^0(\theta,\phi) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}\left(\frac{3z^2-r^2+r^2}{r^2}\right) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}\frac{3z^2}{r^2}$$

which means that z^2 is made of one part $|lm\rangle = |20\rangle$ and $\sqrt{5}/2$ parts $|lm\rangle = |00\rangle$. Therefore

$$|\psi\rangle = N\left(|20\rangle + \frac{\sqrt{5}}{2}|00\rangle\right)$$

Normalizing this means that $N^2(1+5/4) = N^2(9/4) = 1$ so $N = 2/3$. Therefore

$$|\psi\rangle = \frac{2}{3}|20\rangle + \frac{\sqrt{5}}{3}|00\rangle$$

and the probability of measuring $l,m = 2,0$ is $4/9$, the probability of measuring $l,m = 0,0$ is $5/9$, and the probability of measuring any other l,m combination is zero.

Name: ______________________________

PHYS3701 Intro QM I S24 Quiz #11 25 Apr 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A particle of mass m moves in two spatial dimensions x and y under the influence of a potential energy function $V = m\omega^2 r^2/2 = m\omega^2(x^2 + y^2)/2$. Find the three lowest energy distinct eigenvalues and determine the degeneracy of each of them. You don't need to do a formal derivation, but explain your reasoning.

PHYS3701 Intro QM I   Quiz #11   Solution

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The Hamiltonian decouples neatly into two independent simple harmonic oscillators, namely

$$H = \left[\frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2\right] + \left[\frac{p_y^2}{2m} + \frac{1}{2}m\omega^2 y^2\right]$$

Whether you solve this using algebra or differential equations, you will find energy eigenvalues

$$E_{n_x,n_y} = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega = (n_x + n_y + 1)\,\hbar\omega$$

where n_x and n_y are separate non-negative integers. Therefore the energy levels are

E	n_x	n_y	Degeneracy
$\hbar\omega$	0	0	1
$2\hbar\omega$	1	0	
	0	1	2
$3\hbar\omega$	1	1	
	2	0	
	0	2	3