Name: ______________________________

PHYS3701 Intro QM I   S24   Quiz #10   18 Apr 2024

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A system in three dimensional space has a wave function $\psi(\vec{r}) = N f(r)\, z^2$ where r and z are the usual spatial coordinates, $f(r)$ is an arbitrary function, and N is a constant.

(a) Show that the state is a mixture of angular momentum eigenstates $|lm\rangle$ for $l = 0$ and $l = 2$. You should use some table of spherical harmonics from your favorite source.

(b) Find the probabilities that a measurement observes each possible eigenstate $|lm\rangle$.

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The factor z^2 suggests we look at the spherical harmonics with $l = 2$. In fact

$$Y_2^0(\theta,\phi) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}(3\cos^2\theta - 1) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}\frac{3z^2 - r^2}{r^2}$$

Noting that $Y_0^0(\theta,\phi) = 1/\sqrt{4\pi}$, we write

$$Y_2^0(\theta,\phi) + \frac{\sqrt{5}}{2}Y_0^0(\theta,\phi) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}\left(\frac{3z^2 - r^2 + r^2}{r^2}\right) = \frac{\sqrt{5}}{2}\sqrt{\frac{1}{4\pi}}\frac{3z^2}{r^2}$$

which means that z^2 is made of one part $|lm\rangle = |20\rangle$ and $\sqrt{5}/2$ parts $|lm\rangle = |00\rangle$. Therefore

$$|\psi\rangle = N\left(|20\rangle + \frac{\sqrt{5}}{2}|00\rangle\right)$$

Normalizing this means that $N^2(1 + 5/4) = N^2(9/4) = 1$ so $N = 2/3$. Therefore

$$|\psi\rangle = \frac{2}{3}|20\rangle + \frac{\sqrt{5}}{3}|00\rangle$$

and the probability of measuring $l, m = 2, 0$ is $4/9$, the probability of measuring $l, m = 0, 0$ is $5/9$, and the probability of measuring any other l, m combination is zero.