Phys 3701 Intro QM I   Spring 2024   25 Jan 2024

[this class over Zoom. Fingers crossed!]

Review so Far

Hermitian Operators and Basis States

$$A|a'\rangle = a'|a'\rangle \Rightarrow 1 = \sum_{a'} |a'\rangle\langle a'|$$

$$\Leftrightarrow |\alpha\rangle = 1|\alpha\rangle = \sum_{a'} |a'\rangle \underline{\underline{\langle a'|\alpha\rangle}}$$ Coefficients

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Spin-1/2 Example:

$$S_z|\pm\hat{z}\rangle = \pm\frac{\hbar}{2}|\pm\hat{z}\rangle$$

$$|\pm\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle \pm \frac{1}{\sqrt{2}}|-\hat{z}\rangle$$

$$|\pm\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle \pm \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

Building Operators* From Outer Products

$$A = A1 = A\sum_{a'} |a'\rangle\langle a'| = \sum_{a'} a'|a'\rangle\langle a'|$$

* For now, Hermitian Operators

Spin-1/2 Example:

$$S_z = \frac{\hbar}{2}\left[|+\hat{z}\rangle\langle+\hat{z}| - |-\hat{z}\rangle\langle-\hat{z}|\right]$$

$$S_x = \frac{\hbar}{2}\left[|+\hat{z}\rangle\langle-\hat{z}| + |-\hat{z}\rangle\langle+\hat{z}|\right]$$

$$S_y = \frac{\hbar}{2}\left[-i|+z\rangle\langle-\hat{z}| + i|-\hat{z}\rangle\langle+\hat{z}|\right]$$

Today: Matrix Representations

It's how we actually solve many QM problems.

Key point: Pick a basis

$\Rightarrow$ kets, bras, operators are represented in that basis

Different basis, Different values, but same operators/states!!

Simple Example: Spin-1/2 in the $|\pm\hat{z}\rangle$ Basis

$$|\alpha\rangle = \underline{1}\,|\alpha\rangle = \left(|+\hat{z}\rangle\langle+\hat{z}| + |-\hat{z}\rangle\langle-\hat{z}|\right)|\alpha\rangle$$

$$= |+\hat{z}\rangle \underbrace{\langle+\hat{z}|\alpha\rangle} + |-\hat{z}\rangle\underbrace{\langle-\hat{z}|\alpha\rangle}$$

$\Rightarrow$ the two complex numbers $\langle+\hat{z}|\alpha\rangle$ and $\langle-\hat{z}|\alpha\rangle$ completely specify $|\alpha\rangle$ if we agree to use $|\pm\hat{z}\rangle$ basis!

It is handy to write this representation as

$$|\alpha\rangle \doteq \underline{\alpha} = \begin{bmatrix} \langle+\hat{z}|\alpha\rangle \\ \langle-\hat{z}|\alpha\rangle \end{bmatrix}$$ "Column Vector"

Peculiar symbol but this is NOT "="!

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e.g. $|+\hat{z}\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \chi_+$ (in some books)

$|-\hat{z}\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \chi_-$

$$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$ etc...

Similar for bras $\Rightarrow$ Row (Dual) vectors

$$\langle\alpha| = \langle\alpha|1 = \langle\alpha|+\hat{z}\rangle\langle+\hat{z}| + \langle\alpha|-\hat{z}\rangle\langle-\hat{z}|$$
$$= \langle+\hat{z}|\alpha\rangle^*\langle+\hat{z}| + \langle-\hat{z}|\alpha\rangle^*\langle-\hat{z}|$$
$$\doteq \underset{\sim}{\tilde{\alpha}} = \left[\langle+\hat{z}|\alpha\rangle^* \quad \langle-\hat{z}|\alpha\rangle^*\right]$$

etc...

In General

$$|\alpha\rangle = 1|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle \doteq \begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \\ \langle a^{(N)}|\alpha\rangle \end{bmatrix}$$

$$\langle\alpha| = \left[\langle a^{(1)}|\alpha\rangle^* \quad \langle a^{(2)}|\alpha\rangle^* \cdots \langle a^{(N)}|\alpha\rangle^*\right]$$

Operators: Insert Identity twice

$$X = 1X1 = \left(\sum_{a''} |a''\rangle\langle a''|\right) X \left(\sum_{a'} |a'\rangle\langle a'|\right)$$

$$= \sum_{a''}\sum_{a'} |a''\rangle \underbrace{\langle \underset{\text{ROW}}{\underline{a''}}|X|\underset{\text{COLUMN}}{\underline{a'}}\rangle}_{\text{"Matrix Element"}} \langle a'|$$

$$\text{i.e. } X \doteq \underline{\underline{X}} = \begin{bmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots & \langle a^{(1)}|X|a^{(N)}\rangle \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \cdots & \\ \vdots & \vdots & & \end{bmatrix}$$

Operators are represented by a matrix!

NOTE: $$X^\dagger = \sum_{a''}\sum_{a'} |a'\rangle\langle a''|X^\dagger|a'\rangle\langle a''|$$

$$= \sum_{a'}\sum_{a''} |a'\rangle\langle a'|X|a''\rangle^*\langle a''|$$

Hermitian matrix adjoint!

Spin-1/2 Example

$$S_y = \frac{\hbar}{2}\Big[\, |+\hat{z}\rangle (-i) \langle -\hat{z}| + |-\hat{z}\rangle (i) \langle +\hat{z}| \,\Big]$$

$$\doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \underline{S}_y = \frac{\hbar}{2}\,\sigma_y \quad \text{"Pauli Matrices"}$$

NOTE: $\tilde{\underline{S}}_y = \underline{S}_y$ i.e. Hermitian Matrix

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Operators on Kets

Consider $X|\alpha\rangle = |\beta\rangle \Rightarrow \langle a'|X|\alpha\rangle = \langle a'|\beta\rangle$

$$\cancel{\Leftrightarrow \sum_{a''}\sum_{a'} |a''\rangle\langle a''|X|a'\rangle\langle a'| = |\beta\rangle}$$

$$\Leftrightarrow \sum_{a''} \langle a'|X|a''\rangle\langle a''|\alpha\rangle = \langle a'|\beta\rangle$$

matrix multiplication!!

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i.e.
$$\begin{bmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \cdots \\ \vdots & & \ddots \end{bmatrix}\begin{bmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \vdots \end{bmatrix} = \begin{bmatrix} \langle a^{(1)}|\beta\rangle \\ a^{(2)}|\beta\rangle \\ \vdots \end{bmatrix}$$

Similar for Bras, Multiplying Operators

$\Leftrightarrow$ We Can Do All of our "work" using matrix algebra!
[For Discrete Bases !!]

The Eigenvalue Problem

Suppose you want to "solve" the problem

$$B|b'\rangle = b'|b'\rangle$$

for eigenvalues b' and eigenvectors $|b'\rangle$ in terms of some other eivs a' and $|a'\rangle$.

$$\Rightarrow \langle a'|B|b'\rangle = b'\langle a'|b'\rangle$$

$$\sum_{a''} \langle a'|B|a''\rangle\langle a''|b'\rangle = b'\langle a'|b'\rangle$$

i.e. $\underline{\underline{B}}\ \underline{b} = b'\ \underline{b}$ Standard Problem!

Example: $S_y|y'\rangle = y'|y'\rangle$ for spin-1/2

$$\Rightarrow \underline{S_y}\ \underline{y} = y'\ \underline{y}$$

Found $\underline{S_y} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \Rightarrow$ Handy to write $y' = \frac{\hbar}{2}\lambda$

$$* \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \lambda\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

To find λ:

$$\det\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = (-\lambda)^2 - (i)(-i) = \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \pm 1$$

so $y' = \pm\frac{\hbar}{2}$ CORRECT!

Now find eigenvectors.

For $\lambda = +1$: $\cancel{\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}$ * $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

i.e. $-iy_2 = y_1 \qquad iy_1 = y_2$ (same)

$$\Leftrightarrow \ |+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

For $\lambda = -1$:

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{matrix} -iy_2 = -y_1 \\ iy_1 = -y_2 \end{matrix} \text{ or}$$

$$\Leftrightarrow \ |-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

NOTE: this is a consistency check!

i.e. we used these defs of $|\pm\hat{y}\rangle$ to build S_y!

LATER: the physical meaning of angular momentum.