Phys 3701 Intro QM I Spring 2024 2 April 2024

* Still on Zoom from APS Meeting
  ↳ No quiz today
* Can't Do OH tomorrow: Email w/ questions!

Where we Left Off

$$H = \frac{1}{2m}\vec{p}^{\,2} + V(|\vec{r}|)$$ "Central Potential in Relative Coords for equivalent Two-Body Problem"

$$|\vec{r}| = [x^2+y^2+z^2]^{1/2} \equiv r$$ an operator

Rotations in 3D Space

H depends only on $\vec{p}^{\,2}$ and $\vec{r}^{\,2}$

↳ It seems to be "rotationally invariant"

Let's be more formal and see what this means.

"Rotate $|x,y,z\rangle$ about the z-axis"

$$R(\phi,\hat{z})|x,y,z\rangle = |x\cos\phi - y\sin\phi,\ y\cos\phi + x\sin\phi,\ z\rangle$$

Want to relate this to a generator. Angular Momentum ??

↳ "Weyl's Trick": $\phi \to d\phi$

i.e. $$R(d\phi,\hat{z})|x,y,z\rangle = |\underline{x - y\,d\phi},\ \underline{y + x\,d\phi},\ z\rangle$$

Translation in x by $-y\,d\phi$ y by $+x\,d\phi$

i.e. $R(d\phi, \hat{z})\,|x,y,z\rangle = T(-y\,d\phi\,\hat{x})\,T(x\,d\phi\,\hat{y})\,|x,y,z\rangle$

$$= \left[1 - \frac{i}{\hbar} p_x (-y\,d\phi)\right]\left[1 - \frac{i}{\hbar} p_y (x\,d\phi)\right] |x,y,z\rangle$$

$$= \left[1 - \frac{i}{\hbar}(x p_y - y p_x)\,d\phi\right] |x,y,z\rangle \quad + \mathcal{O}(d\phi^2)$$

Recall: $R(d\phi, \hat{z}) = 1 - \frac{i}{\hbar} G\, d\phi$ $\quad G =$ "Generator"

[Hermitian operator]

$\Rightarrow G = x p_y - y p_x = (\vec{r} \times \vec{p})_z = L_z$

"Orbital Angular momentum is the generator of rotations in 3D!"

Note: Choosing the z-axis was an arbitrary choice!

$\Rightarrow$ All this holds for L_x, L_y, too!

i.e. $\vec{L} = \vec{r} \times \vec{p}$ is the generator in 3D!

Invariance

- Will cover this in detail next semester
- For now, see Homework to "get a taste"
- Bottom Line: $R^\dagger H R = H \Rightarrow$ "Invariant"

$\Rightarrow$ Implies that the generator ($\vec{L}$) commutes with the Hamiltonian!

$\Rightarrow$ Simultaneous Eigenvalues !!!

Example: Prove $[L_z, \vec{p}^{\,2}] = 0$ ‖ $[L_z, \vec{r}^{\,2}] = 0$ is similar $\Rightarrow [L_z, H] = 0$ etc...

$$[L_z, \vec{p}^{\,2}] = [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2]$$

Do them one at a time!

$$[L_z, p_x^2] = L_z p_x^2 - p_x L_z p_x + p_x L_z p_x - p_x^2 L_z$$
$$= [L_z, p_x] p_x + p_x [L_z, p_x]$$

$$[L_z, p_x] = [x p_y - y p_x, p_x] = [x, p_x] p_y = i\hbar\, p_y$$

$$\Rightarrow [L_z, p_x^2] = \underline{\underline{2i\hbar\, p_x p_y}}$$

$$[L_z, p_y^2] = [L_z, p_y] p_y + p_y [L_z, p_y]$$
$$[L_z, p_y] = [x p_y - y p_x, p_y] = -[y, p_y] p_x = -i\hbar\, p_x$$
$$\Rightarrow [L_z, p_y^2] = \underline{\underline{-2i\hbar\, p_x p_y}}$$

$[L_z, p_z^2] = [x p_y - y p_x, p_z^2] = 0$ (Everything commutes!)

$\Rightarrow [L_z, \vec{p}^{\,2}] = 0$

- "Rotational symmetry"
- Commuting $\Rightarrow$ Simultaneous Eigenvalues!!

<u>Summary So Far</u>

- $\vec{L} = \vec{r} \times \vec{p}$ generates rotations in 3D space
- $[\vec{L}, H] = 0$ for <u>central potentials</u>: "symmetry"

$\Rightarrow$ Now use what we already know about $\vec{J} \to \vec{L}$!

Central Potentials: Commuting Observables

Our goal: Solve $H|E\rangle = E|E\rangle$ for "central potentials"

We know: $\vec{L}^2 |\ell m\rangle = \ell(\ell+1)\hbar^2 |\ell m\rangle$

$$L_z |\ell m\rangle = m\hbar |\ell m\rangle$$

How? $\vec{L}$ is the generator of (3D) rotations!

Caveat: $\ell = 0, 1, 2, \ldots$ (no half-integers)

[we will see why next week.]

$\Rightarrow$ we search for eigenstates $|E\ell m\rangle$

w/ $H|E\ell m\rangle = E|E\ell m\rangle$

$$\vec{L}^2 |E\ell m\rangle = \ell(\ell+1)\hbar^2 |E\ell m\rangle$$

$$L_z |E\ell m\rangle = m\hbar |E\ell m\rangle$$

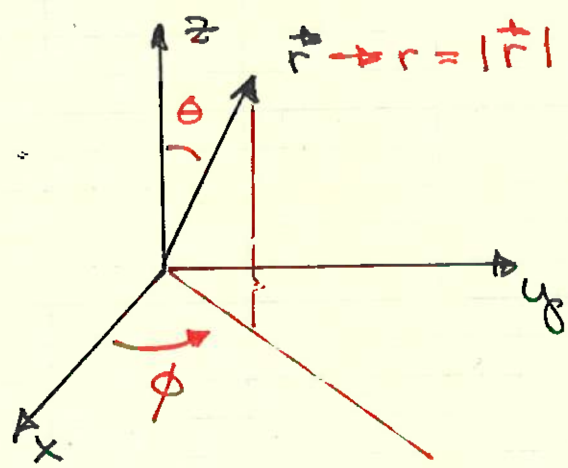
Wave Functions: $\psi_{E\ell m}(\vec{r}) = \langle \vec{r} | E\ell m\rangle$

Spherical coordinates

$\psi_{E\ell m}(r, \theta, \phi)$ makes more sense than $\psi_{E\ell m}(x, y, z)$ because we have spherical symmetry.

Concepts Sec. 4.1.3

"think of r, θ, ϕ in terms of the operators x, y, z"



Recall $H = \frac{1}{2m}\vec{p}^{\,2} + V(r)$ How to connect to $\vec{L}$??

Key (HW): $\vec{L}^2 = (\vec{r}\times\vec{p})\cdot(\vec{r}\times\vec{p})$

$$= \vec{r}^{\,2}\vec{p}^{\,2} - (\vec{r}\cdot\vec{p})^2 + i\hbar\,\vec{r}\cdot\vec{p}$$

$$\Leftrightarrow \langle\vec{r}|\vec{L}^2|\alpha\rangle = \langle\vec{r}|\,[\underline{\vec{r}^{\,2}\vec{p}^{\,2}} - \underline{(\vec{r}\cdot\vec{p})^2} + i\hbar\,\underline{\vec{r}\cdot\vec{p}}\,]\,|\alpha\rangle$$

Do these three one at a time!

$$\langle\vec{r}|\vec{r}^{\,2}\vec{p}^{\,2}|\alpha\rangle = r^2\langle\vec{r}|\vec{p}^{\,2}|\alpha\rangle$$

$$\Leftrightarrow \langle\vec{r}|\vec{p}^{\,2}|\alpha\rangle = \frac{1}{r^2}\langle\vec{r}|\left[(\vec{r}\cdot\vec{p})^2 - i\hbar\,\vec{r}\cdot\vec{p} + \vec{L}^2\right]|\alpha\rangle \quad (*)$$

Our goal in wave-mechanics language is to solve

$$\langle\vec{r}|H|E\rangle = \frac{1}{2m}\langle\vec{r}|\vec{p}^{\,2}|E\rangle + \langle\vec{r}|V(|\vec{r}|)|E\rangle = E\langle\vec{r}|E\rangle$$

$$\text{i.e. } \frac{1}{2m}\underline{\langle\vec{r}|\vec{p}^{\,2}|E\rangle} + V(r)\underbrace{\langle\vec{r}|E\rangle}_{\psi_E(\vec{r})} = E\underbrace{\langle\vec{r}|E\rangle}_{\psi_E(\vec{r})}$$

Next Steps: Substitute this with (*) and $|\alpha\rangle = |E\ell m\rangle$ so

- $\vec{L}^2|E\ell m\rangle = \ell(\ell+1)\hbar^2|E\ell m\rangle$
- write $(\vec{r}\cdot\vec{p})^2$ and $(\vec{r}\cdot\vec{p})$ by using

$$\langle\vec{r}|\vec{p}|\alpha\rangle = \frac{\hbar}{i}\vec{\nabla}\langle\vec{r}|\alpha\rangle$$

in <u>spherical coordinates</u>.

$\Rightarrow$ Differential Equation in Spherical Coordinates (r, θ, ϕ) in terms of E, ℓ (and m, as we'll see)

$\longrightarrow$ Take a Break $\Rightarrow$ Next Week $\longleftarrow$

(If time...) Glimpse of where we are heading.

Hydrogen Atom via Dimensional Analysis

$$H = \frac{1}{2m}\vec{p}^{\,2} - \frac{e^2}{|\vec{r}|}$$ Gaussian Units! (No $1/4\pi\varepsilon_0$)

$$\Rightarrow E = \langle H \rangle = \frac{1}{2m}\langle \vec{p}^{\,2}\rangle - \frac{e^2}{a}$$ a = "size of atom"

Uncertainty principle: $\Delta\vec{p}^{\,2} = \langle \vec{p}^{\,2}\rangle - \underbrace{\langle \vec{p}\rangle^2}_{=0} = \langle \vec{p}^{\,2}\rangle$

But $\Delta\vec{p}^{\,2} = |\Delta\vec{p}|^2 \sim \left(\frac{\hbar}{a}\right)^2$

$$\Rightarrow E \approx \frac{\hbar^2}{2ma^2} - \frac{e^2}{a}$$

- $a \to \infty \Rightarrow$ Larger V, smaller K
- $a \to 0 \Rightarrow$ Smaller V, larger K

Should be a minimum somewhere!

$$\frac{dE}{da} = -\frac{\hbar^2}{ma^3} + \frac{e^2}{a^2} = 0 \Rightarrow a = \frac{\hbar^2}{me^2}$$

$$\Rightarrow E = -\frac{me^4}{2\hbar^2}$$

Expect something like this when we complete the problem.