Phys 3701 Intro QM I Spring 2024 2 April 2024

* Via Zoom from APS 2024
* Homework Posted
* New Syllabus starts today

Quick Review: Wave Mechanics in 1D

Position operator x w/ $x|x\rangle = x|x\rangle$ "operator → eigenvalue"

Translation operator $T(a)|x\rangle = |x+a\rangle$

Normalization $\langle x'|x\rangle = \delta(x-x')$

$\Rightarrow$ $1 = \int dx\, |x\rangle\langle x|$

$T(dx) = 1 - \frac{i}{\hbar} dx\, p$ "Weil's Trick" $\Rightarrow T(a) = e^{-ipa/\hbar}$

p = "momentum" Why?? Well we found that

$H = \frac{1}{2m}p^2 + V(x) \Rightarrow \frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}$ and $\frac{d\langle p\rangle}{dt} = \left\langle -\frac{dV}{dx}\right\rangle$ OK

Define $\psi_\alpha(x) \equiv \langle x|\alpha\rangle \Rightarrow \langle x|p|\alpha\rangle = \frac{\hbar}{i}\frac{\partial \psi_\alpha}{\partial x}$

$\Rightarrow$ $H|E\rangle = E|E\rangle$ Becomes $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x) = E\psi(x)$

"Schrödinger Equation"

Also
- For "Bound states" have $\psi(x \to \pm\infty) = 0$

  $\Rightarrow$ Discrete Energies! (e.g. Box, SHO)
- $\langle x|p|p\rangle = p\langle x|p\rangle$

  and $= \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|p\rangle$ $\Rightarrow \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar}$

Today: Begin Wave Mechanics in 3D

Position and Momentum in 3D

Position Eigenstate $|\vec{r}\rangle = |x,y,z\rangle$

w/ operator $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ $\quad \hat{x},\hat{y},\hat{z}$ unit vectors

i.e. $x|\vec{r}\rangle = x|\vec{r}\rangle \quad y|\vec{r}\rangle = y|\vec{r}\rangle \quad z|\vec{r}\rangle = z|\vec{r}\rangle$

operator $\quad$ eigenvalue

Tacit assumption: x,y,z are commuting operators!

Don't Assume that → "super symmetry" (!)

Normalization

$$\langle\vec{r}|\vec{r}'\rangle = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$
$$\equiv \delta^{(3)}(\vec{r}-\vec{r}')$$

Wave Function $\psi_\alpha(\vec{r}) \equiv \langle\vec{r}|\alpha\rangle$

$$\Rightarrow 1 = \langle\alpha|\alpha\rangle = \int d^3r\,\langle\alpha|\vec{r}\rangle\langle\vec{r}|\alpha\rangle = \int d^3r\,\psi_\alpha^*(\vec{r})\,\psi_\alpha(\vec{r})$$

Translation Operator: Now it has a vector argument

$$T(a_x\hat{x})|x,y,z\rangle = |x+a_x,y,z\rangle$$
$$T(a_y\hat{y})|x,y,z\rangle = |x,y+a_y,z\rangle$$
$$T(a_z\hat{z})|x,y,z\rangle = |x,y,z+a_z\rangle$$

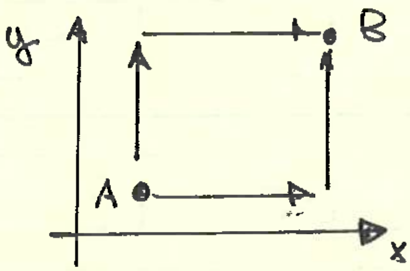
i.e. $T(\vec{a})|\vec{r}\rangle = |\vec{r}+\vec{a}\rangle$

We clearly have

$$T(a_x\hat{x}) = e^{-ip_x a_x/\hbar}$$
$$T(a_y\hat{y}) = e^{-ip_y a_y/\hbar}$$
$$T(a_z\hat{z}) = e^{-ip_z a_z/\hbar}$$

Same argument as for "x" i.e. $dx = \lim_{N\to\infty} \frac{a_x}{N}$ etc...

Important: Translations commute



i.e. $T(a_y\hat{y})T(a_x\hat{x})$
$= T(a_x\hat{x})T(a_y\hat{y})$

$\Rightarrow$ Put $a_x = dx$, $a_y = dy$ and go to second order to find $[p_x, p_y] = 0$ Homework!

"Momentum components commute" $\Rightarrow$ $|p_x, p_y, p_z\rangle = |\vec{p}\rangle$ ok!

This also means that we can write...

$$T(a_x\hat{x})T(a_y\hat{y})T(a_z\hat{z}) = e^{-i(p_x a_x + p_y a_y + p_z a_z)/\hbar}$$
$$= e^{-i\vec{p}\cdot\vec{a}/\hbar}$$

... because the p_x, p_y, p_z all commute!

Back to 1D for a moment

We showed $(xT(\delta x) - T(\delta x)x) = -\frac{i}{\hbar}\delta x\,[x, p_x]$

and $(xT(\delta x) - T(\delta x)x)|\alpha\rangle = \delta x|\alpha\rangle$ $\Rightarrow$ $xT(\delta x) \to (x+\delta x)$

$\Rightarrow$ $[x, p_x] = i\hbar$ "Canonical Commutation Relation"

So it follows directly that

$$[y, p_y] = i\hbar \quad \text{and} \quad [z, p_z] = i\hbar$$

But what about other combination?

i.e. Do y and $T(a_x \hat{x})$ commute?

$$T(a_x \hat{x})\, y\,|\alpha\rangle = T(a_x \hat{x})\, \underset{\text{oper.}}{y} \int d^3r\ |x,y,z\rangle\langle x,y,z|\alpha\rangle$$

$$= T(a_x \hat{x}) \int d^3r\ \underset{\text{eigenvalue}}{y}\,|x,y,z\rangle\langle x,y,z|\alpha\rangle$$

then take it out!

i.e. y $T(\delta x \hat{x})$ has no eigenvalue change!

$$= \int d^3r\, y\, T(a_x \hat{x})\,|x,y,z\rangle\langle x,y,z|\alpha\rangle = y\, T(a_x \hat{x})\,|\alpha\rangle$$

<u>yes</u> $\Longrightarrow$ $[y, p_x] = 0$ etc...

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i.e. $[r_i, p_j] = i\hbar\, \delta_{ij}$ $\qquad r_1, r_2, r_3 = x, y, z$

$\qquad p_1, p_2, p_3 = p_x, p_y, p_z$

<u>Momentum Eigenstates</u> $|p_x, p_y, p_z\rangle = |\vec{p}\rangle$

w/ $\langle \vec{p}|\vec{p}'\rangle = \delta^{(3)}(\vec{p} - \vec{p}')$

And as before $\langle \vec{r}|\vec{p}|\alpha\rangle = \frac{\hbar}{i}\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\langle \vec{r}|\alpha\rangle$

$$= \frac{\hbar}{i}\vec{\nabla}\,\psi_\alpha(\vec{r})$$

and $\langle \vec{r}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\, e^{i\vec{p}\cdot\vec{r}/\hbar}$

Translational Invariance

aka "Translational symmetry"

Consider a two-particle Hamiltonian:

$$H = \frac{1}{2m_1}\vec{p}_1^{\,2} + \frac{1}{2m_2}\vec{p}_2^{\,2} + V(|\vec{r}_1 - \vec{r}_2|)$$

Only depends on difference!

⇒ Expect translational invariance!

Formal: $T_{12}^{\dagger} H T_{12} = H$ (See Homework)

w/ $T_{12} = T_1(\vec{a})T_2(\vec{a}) = e^{-i\vec{P}\cdot\vec{a}/\hbar}$

where $\vec{P} = \vec{p}_1 + \vec{p}_2$ "total momentum"

Find $[\vec{P}, H] = 0$

But $\frac{d}{dt}\langle\vec{P}\rangle = \frac{i}{\hbar}\langle[H, \vec{P}]\rangle = 0$

Recall: $d\langle A\rangle/dt$
$= d/dt\ \langle\alpha;t|A|\alpha;t\rangle$
etc...

⇒ "Total Momentum is Conserved"

We saw the same result in Analytical Mechanics!

"Conservation Laws are Fundamental"

⇒ Much of this physics follows directly from classical mechanics e.g. **Relative and CM Coordinates**

Might not have time for this, but that's ok.

Better operators for two-body physics

$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ "relative"

$\vec{R} \equiv (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1+m_2)$ "CM"

NOTE $[r_i, P_j] = 0$

e.g. $[x_1 - x_2, p_1 + p_2] = [x_1, p_1] - [x_2, p_2] = 0$

Expected from Translational Invariance!

ALSO $[R_i, P_j] = i\hbar\,\delta_{ij}$ (Work it out)

$\Rightarrow$ Define relative momentum $\vec{p} = \dfrac{m_2\vec{p}_1 - m_1\vec{p}_2}{m_1+m_2}$

$\Rightarrow [r_i, p_j] = i\hbar\,\delta_{ij}$ (Work it out)

$\Rightarrow$ we can treat two-body systems just using the relative coordinate and reduced mass.

(As Expected).

But mention this...

Upshot: We will now consider Hamiltonians

$$H = \frac{1}{2m}\vec{p}^{\,2} + V(|\vec{r}|)$$ "Relative $\vec{r}$ and $\vec{p}$ in Two-Body System"

$m = \dfrac{m_1 m_2}{m_1+m_2}$ "Reduced Mass"

"Central Potential"

i.e. Spherically Symmetric