PHYS3701 Intro Quantum Mechanics I HW#11 Due 9 Apr 2024

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Show that the commutativity of infinitesimal x- and y-translations, in other words $[T(dx\,\hat{\mathbf{x}}), T(dy\,\hat{\mathbf{y}})] = 0$, implies that x- and y-momenta commute, that is $[p_x, p_y] = 0$. You will need to carry out the calculation to second order.

(2) A quantum mechanical "symmetry" can be quantified by some unitary operator $\mathcal{S}$ where, for some observable A, $\langle A\rangle$ is unchanged when the state $|\alpha\rangle \to \mathcal{S}|\alpha\rangle$.

(a) Show that this symmetry implies that $[\mathcal{S}, A] = 0$.

(b) Assuming $\mathcal{S} = \mathcal{S}(u)$ where u is continuous, use "Weyl's trick" to write $\mathcal{S}(du)$ in terms of some Hermitian operator $\mathcal{G}$ and show that the symmetry implies that $[\mathcal{G}, A] = 0$.

(c) Illustrate this by showing that the three-dimensional momentum operator $\vec{p}$ is invariant under the translation symmetry operator $T(\vec{a})$. (Don't be worried if it looks like your illustration is trivial. We will study more about symmetries next semester.)

(3) It is reasonable to define a "vector" as a three component object that transforms under rotations just the way you'd expect. Use this definition and the transformation from Problem (2) above to prove the following relationships for a vector operator $\vec{V} = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}}$:

$$[L_z, V_x] = i\hbar V_y \qquad [L_z, V_y] = -i\hbar V_x \qquad [L_z, V_z] = 0$$

You should notice that this definition implies that angular momentum is indeed a vector.

(4) Recall from our Mathematical Physics course, or in the Concepts textbook Eq (4.8), that the totally antisymmetric symbol ϵ_{ijk} has the property that

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

where the summation over 1, 2, and 3 for repeated indices is implied. Writing the components of the orbital angular momentum operator as

$$L_i = \epsilon_{ijk} r_j p_k \qquad \text{where} \qquad r_{1,2,3} = x, y, z$$

and using the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$, show that orbital angular momentum obeys the correct commutation relations for generalized angular momentum that we derived several weeks ago, namely

$$[L_i, L_j] = i\hbar\epsilon_{ijk} L_k$$

(5) Use the techniques from Problem **(4)**, including writing out $[r_i, p_j] = i\hbar\delta_{ij}$ in order to "flip" position and momentum, to prove the relation

$$\vec{L}^2 = \vec{r}^{\,2}\vec{p}^{\,2} - (\vec{r}\cdot\vec{p})^2 + i\hbar\vec{r}\cdot\vec{p}$$