PHYS3701 Intro Quantum Mechanics I HW#7 Due 12 Mar 2024

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A particle of mass m is confined to a three dimensional "infinite box" of side length L in the region $0 \leq x, y, z \leq L$. Solve the time-independent Schrödinger equation to find the energy eigenvalues $E_{n_x,n_y,n_z} = \hbar^2\vec{k}^2/2m$ and eigenfunctions $\psi_{n_x,n_y,n_z}(x,y,z)$ in terms of $\hbar$, m, L, and three positive integers n_x, n_y, and n_z. (This is easy to do using the technique of separation of variables and following what we did for the one-dimensional case.) Make a table of the lowest three energy levels, including their *degeneracy*, that is the number of combinations (n_x, n_y, n_z) that give the same energy.

(2) Write the operator S_x for a spin-1/2 system as linear combination of outer products of the $|\pm\hat{\mathbf{z}}\rangle$ and show that its rotation $\exp(+i\phi S_z/\hbar)S_x\exp(-i\phi S_z/\hbar)$ is just what you expect. (This is written as the transformation of an operator. You might prefer to think of this in terms of the expectation value of the rotated operator in some state.)

(3) The three Pauli spin matrices are given by

$$\underline{\underline{\sigma_x}} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{\underline{\sigma_y}} \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \underline{\underline{\sigma_z}} \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) Show that the representation of the spin operator $\vec{S}$ in the $|\pm\hat{\mathbf{z}}\rangle$ basis can be written as $\underline{\underline{\vec{S}}} = (\hbar/2)\underline{\underline{\vec{\sigma}}}$. (You are free to use results we have derived in class or on prior homework.)

(b) Prove that $(\underline{\underline{\vec{\sigma}}} \cdot \vec{a})^2 = |\vec{a}|^2\,\underline{\underline{1}}$ where the components of $\vec{a}$ are real.

(c) Show that the rotation operator for spin-1/2 systems can be represented in the $|\pm\hat{\mathbf{z}}\rangle$ basis as

$$\underline{\underline{\mathcal{D}}}^{(1/2)}(\hat{n},\phi) = \exp\left[-\frac{i}{\hbar}\underline{\underline{\vec{S}}}\cdot\hat{n}\phi\right] = \underline{\underline{1}}\cos\left(\frac{\phi}{2}\right) - i\underline{\underline{\vec{\sigma}}}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)$$

(d) Repeat Problem **(2)** above using matrix representations of the operators.

(4) Construct the matrix representations of the operators J_x and J_y for a spin-one system, in the J_z basis, spanned by the kets $|+\rangle \equiv |1,1\rangle$, $|0\rangle \equiv |1,0\rangle$, and $|-\rangle \equiv |1,-1\rangle$. Use these matrices to find the three analogous eigenstates for each of the two operators J_x and J_y in terms of $|+\rangle$, $|0\rangle$, and $|-\rangle$. You are welcome to use MATHEMATICA or some other app to find the eigenvalues and eigenstates after you've constructed the matrices.

(5) Using the fact that J_x, J_y, J_z, and $J_\pm \equiv J_x \pm iJ_y$ satisfy the usual angular-momentum commutation relations, prove that

$$\vec{J}^2 \equiv J_x^2 + J_y^2 + J_z^2 = J_z^2 + J_+J_- - \hbar J_z$$

Using this result, or otherwise, derive the coefficient c_- that appears in $J_-|jm\rangle = c_-|j,m-1\rangle$.