Tips for Homework 12 Problem 2 PHYS 3701 Spring 2024

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This is a nice problem, not hard but with a lot of physics in it. It brings together many things that you have already learned, so let me remind you of these to help you understand and solve the problem.

You are given a wave function

$$\psi(\vec{r}) = Nf(r)(x + y + 2z)$$

and your job is to figure out what are the possible results of a measurement of L_z and $\vec{L}^2$ and the probabilities for getting those results. You are asked to start by finding the normalization constant N.

In the beginning you learned that any observable corresponds to some Hermitian operator A, which has eigenstates $|a'\rangle$ so that $A|a'\rangle = a'|a'\rangle$. We postulated that the only possible result of a measurement of A is one of the eigenvalues a', and that the probability for obtaining that result if the system is in an arbitrary state $|\alpha\rangle$ is $|\langle a'|\alpha\rangle|^2$, assuming that the state $|\alpha\rangle$ has been properly normalized so that $\langle\alpha|\alpha\rangle = 1$. We went on to show that the $|a'\rangle$ form a complete, orthonormal set, so that for any state $|\alpha\rangle$,

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$$

which means that the probability of measuring a' is also given by $|c_{a'}|^2$.

You also learned that for states that can be represented in the position basis $|\vec{r}'\rangle$, we can define a "wave function" for an arbitrary state $|\alpha\rangle$ as

$$\psi_\alpha(\vec{r}') \equiv \langle\vec{r}'|\alpha\rangle \qquad \text{with} \qquad 1 = \langle\alpha|\alpha\rangle = \int d^3r'\, \langle\alpha|\vec{r}'\rangle\langle\vec{r}'|\alpha\rangle = \int d^3r'\, \psi^*_\alpha(\vec{r}')\psi_\alpha(\vec{r}')$$

which, of course, tells you how to find the value of N in the homework problem.

For the homework problem, you are concerned with the Hermitian operators L_z and $\vec{L}^2$ and their eigenstates $|a'\rangle = |lm\rangle$. We call the wave functions of these eigenstates spherical harmonics and write $Y_l^m(\theta,\phi) = \langle\hat{r}'|lm\rangle$. You can easily look up these functions in a textbook, online, or using Mathematica.

Therefore, you can find the probabilities in this homework problem, if you want, from

$$\langle lm|\alpha\rangle = \int d^3r\, \langle lm|\vec{r}\rangle\langle\vec{r}|\alpha\rangle = \int d^3r\, \left(Y_l^m(\theta,\phi)\right)^* \psi(\vec{r})$$

where I'm not bothering to write "primes" anymore. In principle, you could do this by trying a whole bunch of spherical harmonics until you come up with probabilities that add up to unity, but there is an easier way if you look carefully at the given wave function.

The key is to notice the different powers of x, y, and z in the wave function, and realize that the spherical harmonics contain factors of sines and cosines of θ and ϕ, which themselves can be written in terms of x/r, y/r, and z/r. In fact, many tables of spherical harmonics are written this way. This makes it easy to suss out the values of c_{lm} in the wave function.

Problem Solution

We'll be formal about this at first, but then we'll show a simple shortcut. Normalizing is easy, that is use MATHEMATICA or something to carry out the integral in

$$\int d^3r\, \psi^*(\vec{r})\psi(\vec{r}) = 1$$

and use this to solve for N. Now we can write the state as

$$|\psi\rangle = \sum_l \sum_m c_{l'm'}|\alpha l'm'\rangle$$

where "α" stands for whatever linear combination of energy eigenstates gives this state. The $|c_{lm}|^2$ give the probabilities we are looking for. These are easy to find from

$$c^*_{lm} = \langle\psi|\alpha lm\rangle = \int d^3r\, \psi^*(\vec{r})g(r)Y_l^m(\theta,\phi)$$

where $g(r)$ is the radial part of the wave function. For this problem, it is easy to see that

$$x + y + 2z = r\sqrt{\frac{2\pi}{3}}\left[(1+i)Y_1^{-1}(\theta,\phi) + (1-i)Y_1^1(\theta,\phi) - \frac{4}{\sqrt{2}}Y_1^0(\theta,\phi)\right]$$

and doing the integrals gives you $|c_{1,1}|^2 = 1/6 = |c_{1,-1}|^2$ and $|c_{1,0}|^2 = 2/3$, which add up to unity. All the other c_{lm} are zero, of course, because of the orthogonality of the $Y_l^m(\theta,\phi)$.

Now the simple way to get the result is to realize that x and y are both equal parts Y_1^1 and Y_1^{-1}, while z is "all" Y_1^0. Therefore $x+y+2z$ is one part $m=1$, one part $m=-1$ and two parts $m=0$. That is, the three coefficients of the spherical harmonics are, effectively, 1, 1, and 2, respectively. The sum of these squares is 6, so the probabilities should be 1/6, 1/6, and $4/6 = 2/3$, respectively.