Name: ____________________

PHYS3101 Analytical Mechanics   S23   Quiz #1   31 Aug 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

An object of mass m is fired vertically upward from the Earth's surface with speed v_0. It experiences a linear drag force with magnitude bv where b is a constant and v is the velocity of the object. Assuming that the Earth's gravity exerts a constant force throughout the flight, find the time T at which the object reaches its maximum height, in terms of m, b, v_0, and the acceleration g from gravity.

PHY3101 Analytical Mechanics Quiz #1 Solution

An object of mass m is fired vertically upward from the Earth's surface with speed v_0. It experiences a linear drag force with magnitude bv where b is a constant and v is the velocity of the object. Assuming that the Earth's gravity exerts a constant force throughout the flight, find the time T at which the object reaches its maximum height, in terms of m, b, v_0, and the acceleration g from gravity.

See Sec.2.2 in Taylor or Sec.3.2.4 in Concepts. The equation of motion is

$$m\dot{v} = m\frac{dv}{dt} = -mg - bv \qquad \text{with} \qquad v = v_0 \text{ at } t = 0 \qquad \text{and} \qquad v = 0 \text{ at } t = T$$

The rest is straightforward algebra and integration:

$$\begin{aligned}
dv &= -g\left(1 + \frac{b}{mg}v\right) dt \\
\int_{v_0}^{0} \frac{dv}{1 + (b/mg)v} &= -g\int_0^T dt \\
\frac{mg}{b}\log\left(1 + \frac{b}{mg}v\right)\Bigg|_{v_0}^{0} &= -gT \\
T &= \frac{m}{b}\log\left(1 + \frac{b}{mg}v_0\right)
\end{aligned}$$

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #2 7 Sep 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

An object of mass m is attracted to the origin by a force $F = -k/r^2$ where k is a constant, and r is the distance to the origin. Let r and ϕ be the usual polar coordinates in a plane.

(a) Find the potential energy function $U(r)$ where $U(r \to \infty) = 0$.

(b) Construct the Lagrangian $\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$.

(c) Show that the quantity $\ell \equiv mr^2\dot{\phi}$ is a constant in time.

(d) Find the differential equation of motion for r in terms of m, k, and ℓ.

PHY3101 Analytical Mechanics Quiz #2 Solution

An object of mass m is attracted to the origin by a force $F = -k/r^2$ where k is a constant, and r is the distance to the origin. Let r and ϕ be the usual polar coordinates in a plane.

(a) Find the potential energy function $U(r)$ where $U(r \to \infty) = 0$.
(b) Construct the Lagrangian $\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$.
(c) Show that the quantity $\ell \equiv mr^2\dot{\phi}$ is a constant in time.
(d) Find the differential equation of motion for r in terms of m, k, and ℓ.

The potential energy function is a familiar result. Using (4.13) in Taylor, with $r_0 = \infty$,

$$U(r) = -\int_\infty^r F(r')\,dr' = \int_\infty^r \frac{k}{r'^2}\,dr' = \left[-\frac{k}{r'}\right]_\infty^r = -\frac{k}{r}$$

For the Lagrangian, use the kinetic energy $m\mathbf{v}^2/2$ in polar coordinates (1.43) in Taylor so

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} - U(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{k}{r}$$

The Lagrange equation for ϕ is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = 0 \qquad \text{so} \qquad mr^2\dot{\phi} = \text{constant} \equiv \ell$$

The Lagrange equation for r is

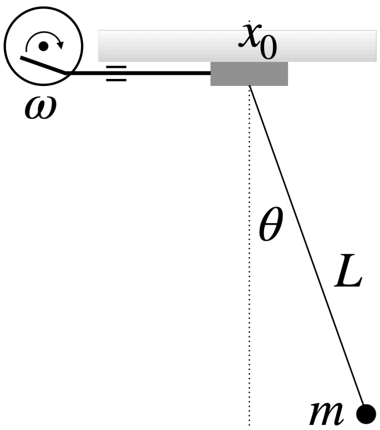
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r} \qquad \text{so} \qquad m\ddot{r} = mr\dot{\phi}^2 - \frac{k}{r^2} = \frac{\ell^2}{mr^3} - \frac{k}{r^2}$$

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #3 14 Sep 2023

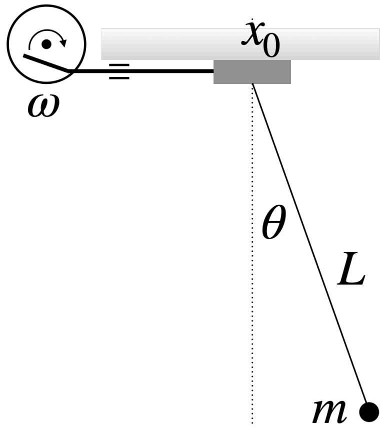
You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.



A pendulum made from a string of length L and bob mass m swings in the vertical plane through an angle θ. It hangs under gravity near the Earth's surface. The top of the string is fixed vertically but moves horizontally with a position $x_0(t) = A \sin \omega t$. Find the (differential) equation of motion for the angle $\theta(t)$. Show that the answer is what you expect if $\omega = 0$.

PHY3101 Analytical Mechanics Quiz #3 Solution



A pendulum made from a string of length L and bob mass m swings in the vertical plane through an angle θ. It hangs under gravity near the Earth's surface. The top of the string is fixed vertically but moves horizontally with a position $x_0(t) = A\sin\omega t$. Find the (differential) equation of motion for the angle $\theta(t)$. Show that the answer is what you expect if $\omega = 0$.

This problem is very similar to the homework problem with the elevator accelerating upward, but now the pivot point of the pendulum is moving sideways instead of upward, and you are explicitly given its motion.

The position of the mass, with $y = 0$ at the bottom of the pendulum swing, is given by

$$\begin{aligned} x &= x_0 + L\sin\theta = A\sin\omega t + L\sin\theta \\ y &= L(1-\cos\theta) \end{aligned}$$

The velocities are

$$\begin{aligned} \dot{x} &= \omega A\cos\omega t + L\dot{\theta}\cos\theta \\ \dot{y} &= L\dot{\theta}\sin\theta \end{aligned}$$

The Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy \\ &= \frac{1}{2}m(L^2\dot{\theta}^2 + \omega^2A^2\cos^2\omega t + 2\omega AL\dot{\theta}\cos\omega t\cos\theta) + mgL\cos\theta + \text{constant} \end{aligned}$$

The equation of motion is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} &= \frac{d}{dt}\left[mL^2\dot{\theta} + m\omega AL\cos\omega t\cos\theta\right] + m\omega AL\dot{\theta}\cos\omega t\sin\theta + mgL\sin\theta \\ &= mL^2\ddot{\theta} - m\omega^2AL\sin\omega t\cos\theta - m\omega AL\dot{\theta}\cos\omega t\sin\theta \\ &\quad + m\omega AL\dot{\theta}\cos\omega t\sin\theta + mgL\sin\theta \\ &= mL^2\ddot{\theta} - m\omega^2AL\sin\omega t\cos\theta + mgL\sin\theta = 0 \end{aligned}$$

This simplifies to

$$\ddot{\theta} + \omega_0^2\sin\theta = \frac{A}{L}\omega^2\sin\omega t\cos\theta \qquad \text{where} \qquad \omega_0^2 \equiv \frac{g}{L}$$

is the natural frequency of the simple pendulum. For $\omega = 0$ this is just the equation of motion for a simple pendulum, as expected. Notice that the result simplifies to a forced, undamped harmonic oscillator for $\theta \ll 1$.

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #4 21 Sep 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A mass m is tied by a massless string to a point on a horizontal, frictionless table. The mass is executing circular motion with radius R at constant speed $v = R\dot{\phi}$. Assume there are perpendicular x and y axes with origin at the fixed point of the string. Use the method of Lagrange multipliers to find the constraint forces, and show that this is what you expect by equating the string tension to the centripetal acceleration of the mass.

PHY3101 Analytical Mechanics Quiz #4 Solution

A mass m is tied by a massless string to a point on a horizontal, frictionless table. The mass is executing circular motion with radius R at constant speed $v = R\dot{\phi}$. Assume there are perpendicular x and y axes with origin at the fixed point of the string. Use the method of Lagrange multipliers to find the constraint forces, and show that this is what you expect by equating the string tension to the centripetal acceleration of the mass.

Write the constraint as $f(x, y) = x^2 + y^2 = R^2$. There is no potential energy term, so

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

The modified Lagrange equations are then

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 2\lambda x - m\ddot{x} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial y} + \lambda\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 2\lambda y - m\ddot{y} = 0$$

The constraints are realized with the parameterization $x = R\cos\phi$ and $y = R\sin\phi$ so

$$\begin{array}{rlcl} & \dot{x} = -R\dot{\phi}\sin\phi & \text{so} & \ddot{x} = -R\dot{\phi}^2\cos\phi - R\ddot{\phi}\sin\phi \\ \text{and} & \dot{y} = R\dot{\phi}\cos\phi & \text{so} & \ddot{y} = -R\dot{\phi}^2\sin\phi + R\ddot{\phi}\cos\phi \end{array}$$

The two modified Lagrange equations become

$$\begin{array}{rrcl} & 2\lambda R\cos\phi + mR\dot{\phi}^2\cos\phi + mR\ddot{\phi}\sin\theta & = & 0 \\ \text{and} & 2\lambda R\sin\phi + mR\dot{\phi}^2\sin\phi - mR\ddot{\phi}\cos\phi & = & 0 \end{array}$$

Multiply the first equation by $\cos\phi$ and the second by $\sin\phi$ and add to get

$$2\lambda = -m\dot{\phi}^2$$

The constraint forces are therefore

$$F_x = \lambda\frac{\partial f}{\partial x} = 2\lambda x = -m\dot{\phi}^2 R\cos\phi \qquad \text{and} \qquad F_y = \lambda\frac{\partial f}{\partial y} = 2\lambda y = -m\dot{\phi}^2 R\sin\phi$$

From centripetal acceleration considerations, the tension (which is negative in both x and y components in the first quadrant) is given by

$$T = m\frac{v^2}{R} = mR\dot{\phi}^2$$

so the analysis above indeed gives the correct x and y components.

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #5 28 Sep 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

We showed in class that the energy of a mass μ in a Kepler orbit is a constant given by

$$E = \frac{1}{2}\mu\dot{r}^2 - \frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

for a central potential $U(r) = -\gamma/r$ and (constant) angular momentum ℓ. By considering the motion of the mass at its points of minimum or maximum approach to the origin (aka the "focus"), derive an expression for E in term of γ, μ, ℓ, and the orbit's eccentricity ϵ. Show that $E < 0$ if $\epsilon < 1$ and $E > 0$ if $\epsilon > 1$.

PHY3101 Analytical Mechanics Quiz #5 Solution

We showed in class that the energy of a mass μ in a Kepler orbit is a constant given by

$$E = \frac{1}{2}\mu\dot{r}^2 - \frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

for a central potential $U(r) = -\gamma/r$ and (constant) angular momentum ℓ. By considering the motion of the mass at its points of minimum or maximum approach to the origin (aka the "focus"), derive an expression for E in term of γ, μ, ℓ, and the orbit's eccentricity ϵ. Show that $E < 0$ if $\epsilon < 1$ and $E > 0$ if $\epsilon > 1$.

See Taylor page 313. The radial speed $\dot{r} = 0$ at the points of either closest or farthest approach. These distances are given by

$$r_{\text{min}} = \frac{c}{1+\epsilon} \qquad \text{and} \qquad r_{\text{max}} = \frac{c}{1-\epsilon} \qquad \text{i.e.} \qquad r_{\text{min,max}} = \frac{c}{1\pm\epsilon} \qquad \text{where} \qquad c = \frac{\ell^2}{\gamma\mu}$$

Plugging these into the equation for the energy (with $\dot{r} = 0$) gives

$$\begin{aligned}
E &= -\frac{\gamma}{c}(1\pm\epsilon) + \frac{\ell^2}{2\mu c^2}(1\pm\epsilon)^2 \\
&= -\frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon) + \frac{\ell^2\gamma^2\mu^2}{2\mu\ell^4}(1\pm\epsilon)^2 \\
&= -\frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon) + \frac{\gamma^2\mu}{2\ell^2}(1\pm\epsilon)^2 \\
&= \frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon)\left[-1 + \frac{1}{2}(1\pm\epsilon)\right] \\
&= \frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon)\left[-1 + \frac{1}{2} \pm \frac{\epsilon}{2}\right] \\
&= \frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon)(-1)\left[\frac{1}{2} \mp \frac{\epsilon}{2}\right] \\
&= -\frac{\gamma^2\mu}{2\ell^2}(1\pm\epsilon)(1\mp\epsilon) = -\frac{\gamma^2\mu}{2\ell^2}(1-\epsilon^2)
\end{aligned}$$

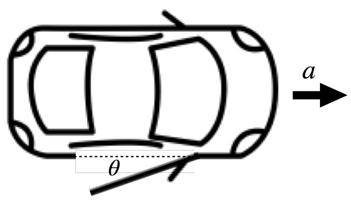
which agrees with Taylor (8.58). This makes it trivial to show that $E < 0$ if $\epsilon < 1$ and $E > 0$ if $\epsilon > 1$, consistent with Taylor Fig.8.4 for elliptical ($\epsilon < 1$) and hyperbolic ($\epsilon > 1$) orbits.

Name: ____________________

PHYS3101 Analytical Mechanics S23 Quiz #6 5 Oct 2023

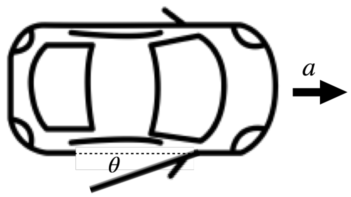
You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.



A car on a flat, horizontal street accelerates from rest with an acceleration $a = 2$ m/s^2. One of the doors is initially open an angle $\theta \ll 1$. The door is 1 m long and has a mass $m = 20$ kg. Assuming the door is of uniform density, find the amount of time it takes for the door to close.

PHY3101 Analytical Mechanics Quiz #6 Solution



A car on a flat, horizontal street accelerates from rest with an acceleration $a = 2$ m/s^2. One of the doors is initially open an angle $\theta \ll 1$. The door is 1 m long and has a mass $m = 20$ kg. Assuming the door is of uniform density, find the amount of time it takes for the door to close.

This is a pendulum problem, except that instead of gravity ("mg") pulling downward on the mass, there is a pseudo force ma acting towards the left, in the car's frame of reference. The door's mass is irrelevant, but since the density is uniform we can take the length of the pendulum to be one half of the door's length. The door closes in one-fourth of a period, so

$$\text{time} = \frac{1}{4}2\pi\sqrt{\frac{\ell}{a}} = \frac{1}{4}2\pi\sqrt{\frac{1/2 \text{ m}}{2 \text{ m/s}^2}} = \frac{\pi}{4} \text{ sec}$$

The solution above is incorrect! The period of a physical pendulum is not equal to that of a point mass pendulum with all the mass at the CM. This does not take into account the forces that keep the rod rigid. Following is the correct derivation for a physical pendulum.

Analyze a stiff rod of length L and mass M in terms of the torque from gravity on the CM about the pivot point. That is, the torque $\tau = I\alpha$ where $I = ML^2/3$, $\alpha = \ddot{\theta}$, and $\tau = -(MgL/2)\sin\theta$. Then

$$-\frac{1}{2}MgL\sin\theta = \frac{1}{3}ML^2\ddot{\theta} \qquad \text{so} \qquad \ddot{\theta} = -\frac{2}{3}\frac{g}{L}\sin\theta \approx -\frac{2}{3}\frac{g}{L}\theta \equiv -\omega^2\theta$$

for $\theta \ll 1$. Therefore, the time it takes the car door to close is

$$\text{time} = \frac{1}{4}2\pi\sqrt{\frac{3}{2}\frac{\ell}{a}} = \frac{\pi}{4}\sqrt{\frac{3}{2}} \text{ sec}$$

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #7 12 Oct 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Calculate the inertia tensor for a uniform sphere of mass M and radius R for a coordinate system with origin located at the center of the sphere, and then find the principle moments of inertia. You may define the x, y, and z axes any way you want, so long as they are orthogonal and right handed.

PHY3101 Analytical Mechanics Quiz #7 Solution

Calculate the inertia tensor for a uniform sphere of mass M and radius R for a coordinate system with origin located at the center of the sphere, and then find the principle moments of inertia. You may define the x, y, and z axes any way you want, so long as they are orthogonal and right handed.

The sphere is 100% symmetric, so the inertia tensor is already diagonal and all three diagonal elements are the same. The result is well known, but also very easy to derive, using spherical coordinates, namely (using the substitution $\mu = \cos\theta$)

$$\begin{aligned}
\lambda &= \int_{\text{Sphere}} \rho^2\, dm = \frac{M}{4\pi R^3/3} \int_{\text{Sphere}} \rho^2\, dV \\
&= \frac{3M}{4\pi R^3} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^R (r\sin\theta)^2\, r^2 dr \\
&= \frac{3M}{2R^3} \int_0^{\pi} \sin^2\theta\, \sin\theta\, d\theta \int_0^R r^2\, r^2 dr \\
&= \frac{3M}{2R^3} \int_{-1}^{1} (1-\mu^2)\, d\mu\, \frac{R^5}{5} \\
&= \frac{3}{10} M R^2 \left[\mu - \frac{1}{3}\mu^3\right]_{-1}^{1} \\
&= \frac{3}{10} M R^2 \frac{4}{3} \\
&= \frac{2}{5} M R^2
\end{aligned}$$

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #8 19 Oct 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A flat square plate of mass M and side length L rotates with angular velocity ω about an axis perpendicular to the plane of the plate and passing through one corner. Find the kinetic energy of the plate in terms of M, L, and ω. You are welcome to use your homework or a Google search to find the moment of inertia of the plate, but make it clear from where you get your answer.

PHY3101 Analytical Mechanics Quiz #8 Solution

A flat square plate of mass M and side length L rotates with angular velocity ω about an axis perpendicular to the plane of the plate and passing through one corner. Find the kinetic energy of the plate in terms of M, L, and ω. You are welcome to use your homework or a Google search to find the moment of inertia of the plate, but make it clear from where you get your answer.

The kinetic energy is $I\omega^2/2$ where $I = I_{\mathrm{zz}}$ is the component of the inertia tensor in a coordinate system where the plate is in the xy plane, and the z-axis passes through the origin. For your homework, you found for a rectangular plate of sides a and b,

$$I_{zz} = \frac{M}{3}(a^2 + b^2) = \frac{2}{3}ML^2$$

I confirmed this answer by Googling "moment of inertia plate corner" and I found

https://www.toppr.com/ask/en-us/question/the-moment-of-inertia-of-a-thin-uniform-rectangular-plate/#

It is also not difficult to find the moment of inertia through the symmetry axis (which passes through the center of mass), namely $ML^2/6$. From the parallel axis theorem, then, the moment of inertia about a corner is this plus $M\Delta^2$ where $\Delta = L\sqrt{2}/2$, that is

$$I_z z = \frac{1}{6}ML^2 + M\frac{2L^2}{4} = \frac{1}{12}ML^2(2+6) = \frac{2}{3}ML^2$$

In any case, the kinetic energy is

$$T = \frac{1}{2}\frac{2}{3}ML^2\,\omega^2 = \frac{1}{3}ML^2\omega^2$$

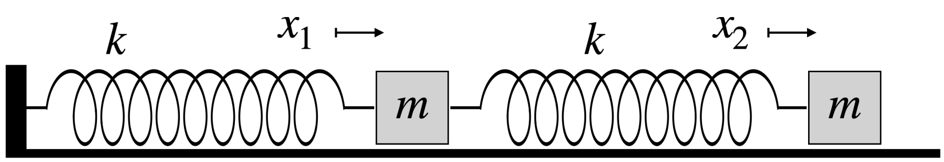
Name: ______________________

PHYS3101 Analytical Mechanics S23 Quiz #9 26 Oct 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Two masses m are connected to each other and to a fixed wall by two identical springs k:



(a) Construct the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$.

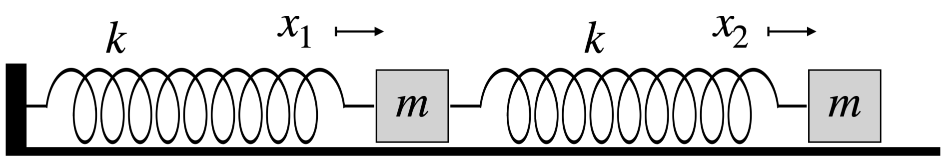
(b) Find the equations of motion for $\ddot{x}_1$ and $\ddot{x}_2$.

(c) Find the matrices $\underline{\underline{M}}$ and $\underline{\underline{K}}$ to write the equations of motion as $\underline{\underline{M}}\,\underline{\ddot{x}} = -\underline{\underline{K}}\,\underline{x}$.

(d) Find the eigenfrequencies of this system.

PHY3101 Analytical Mechanics Quiz #9 Solution

Two masses m are connected to each other and to a fixed wall by two identical springs k:



(a) Construct the Lagrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$.
(b) Find the equations of motion for $\ddot{x}_1$ and $\ddot{x}_2$.
(c) Find the matrices $\underline{\underline{M}}$ and $\underline{\underline{K}}$ to write the equations of motion as $\underline{\underline{M}}\,\underline{\ddot{x}} = -\underline{\underline{K}}\,\underline{x}$.
(d) Find the eigenfrequencies of this system.

The Lagrangian is straightforward.

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2$$

The Euler-Lagrange equations give us the equations of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial \mathcal{L}}{\partial x_1} \quad \text{so} \quad m\ddot{x}_1 = -kx_1 - k(x_2 - x_1)(-1) = -2kx_1 + kx_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{\partial \mathcal{L}}{\partial x_2} \quad \text{so} \quad m\ddot{x}_2 = -k(x_2 - x_1)(-1) = kx_1 - kx_2$$

These can be written as $\underline{\underline{M}}\,\underline{\ddot{x}} = -\underline{\underline{K}}\,\underline{x}$ where

$$\underline{\underline{M}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad \underline{\underline{K}} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

The eigenfrequencies ω^2 are determined from

$$\left|\underline{\underline{K}} - \omega^2\underline{\underline{M}}\right| = \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{vmatrix} = m\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$

where $\omega_0^2 \equiv k/m$. The characteristic equation is therefore

$$(2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

The two solutions for ω^2 are just from the quadratic equation, that is

$$\omega^2 = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2} = \frac{3 \pm \sqrt{5}}{2}\omega_0$$

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #10 2 Nov 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A mass m is confined to a long straight rod rotating with fixed angular velocity ω in the horizontal plane. Using the coordinate r measured from the center of the rod,

(a) Construct the Lagrangian $\mathcal{L}(r, \dot{r})$.

(b) Find the conjugate momentum p.

(c) Construct the Hamiltonian $\mathcal{H}(r, p)$.

(d) Determine Hamilton's equations of motion for the mass.

(e) **Extra Credit:** Describe the motion of the mass for $r(0) > 0$ and $p(0) = 0$.

PHY3101 Analytical Mechanics Quiz #10 Solution

A mass m is confined to a long straight rod rotating with fixed angular velocity ω in the horizontal plane. Using the coordinate r measured from the center of the rod,

(a) Construct the Lagrangian $\mathcal{L}(r, \dot{r})$.
(b) Find the conjugate momentum p.
(c) Construct the Hamiltonian $\mathcal{H}(r, p)$.
(d) Determine Hamilton's equations of motion for the mass.
(e) **Extra Credit:** Describe the motion of the mass for $r(0) > 0$ and $p(0) = 0$.

$$\begin{aligned}
\mathcal{L}(r, \dot{r}) &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \\
\mathcal{H}(r, p) &= p\dot{r} - \mathcal{L} = p\frac{p}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 - \frac{1}{2}m\omega^2 r^2 = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 r^2 \\
\dot{r} &= \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \\
\dot{p} &= -\frac{\partial \mathcal{H}}{\partial r} = m\omega^2 r
\end{aligned}$$

Combining Hamilton's equations gives

$$\ddot{r} = \omega^2 r \qquad \text{so} \qquad r(t) = Ae^{\omega t} + Be^{-\omega t}$$

Setting $\dot{p}(0) = m\omega(A - B) = 0$ means $A = B$ so $r(t)$ is proportional to $\cosh(\omega t)$. If $r(0) > 0$ then the coefficient is nonzero and the mass flies off on the rod exponentially. This, of course, agrees perfectly with what you'd expect.

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #11 9 Nov 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Two particles A and B with four-momenta p_A and p_B collide. Show that the total energy E_{CM} in the center-of-mass frame is given by

$$E_{\text{CM}}^2 = (p_A + p_B)^2 c^2$$

PHY3101 Analytical Mechanics Quiz #11 Solution

Two particles A and B with four-momenta p_A and p_B collide. Show that the total energy E_{CM} in the center-of-mass frame is given by

$$E_{\text{CM}}^2 = (p_A + p_B)^2 c^2$$

Write the four-momenta in the CM frame as

$$p_A = \left(\frac{E_A}{c}, \vec{p}_A\right) \qquad \text{and} \qquad p_B = \left(\frac{E_B}{c}, \vec{p}_B\right)$$

where the definition of the CM frame means that $\vec{p}_A + \vec{p}_B = 0$. Then

$$(p_A + p_B)^2 = \left(\frac{E_A + E_B}{c}, \vec{p}_A + \vec{p}_B\right)^2 = \left(\frac{E_A + E_B}{c}, \vec{0}\right)^2 = \frac{(E_A + E_B)^2}{c^2}$$

However, $E_A + E_B = E_{\text{CM}}$, so

$$E_{\text{CM}}^2 = (p_A + p_B)^2 c^2$$

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #12 16 Nov 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A slice of raisin bread with area A has N raisins embedded in it. Model each raisin as a sphere of radius R. If you fire 1000 BB's from a pellet gun normal to the plane of the bread slice, and randomly spread over its area, then, on average and in terms of A, N, and R, how many raisins would you expect to hit? You can assume that there are few enough raisins so that none of them overlap, and that the BB's have negligible size.

PHY3101 Analytical Mechanics Quiz #12 <u>Solution</u>

A slice of raisin bread with area A has N raisins embedded in it. Model each raisin as a sphere of radius R. If you fire 1000 BB's from a pellet gun normal to the plane of the bread slice, and randomly spread over its area, then, on average and in terms of A, N, and R, how many raisins would you expect to hit? You can assume that there are few enough raisins so that none of them overlap, and that the BB's have negligible size.

The area covered by raisins is $N \times \pi R^2$, and the probability of hitting a raisin is just this area divided by the total area of the bread slice, so you expect, on average,

$$n_{\text{hit}} = 1000 \times \frac{N\pi R^2}{A}$$

Name: ______________________________

PHYS3101 Analytical Mechanics S23 Quiz #13 7 Dec 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A projectile of mass m flies through the air without drag close to the Earth's surface. Using a coordinate system where x is horizontal and y is upward, write down the Hamiltonian function $\mathcal{H}(x, y, p_x, p_y)$ and show that Hamilton's equations reduce to the correct second order differential equations for $x(t)$ and $y(t)$.

PHY3101 Analytical Mechanics Quiz #13 Solution

A projectile of mass m flies through the air without drag close to the Earth's surface. Using a coordinate system where x is horizontal and y is upward, write down the Hamiltonian function $\mathcal{H}(x, y, p_x, p_y)$ and show that Hamilton's equations reduce to the correct second order differential equations for $x(t)$ and $y(t)$.

There are no explicit time dependences, so the Hamiltonian is just the total energy.

$$\mathcal{H}(x, y, p_x, p_y) = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy$$

Hamilton's equations for x are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m} \qquad \text{and} \qquad \dot{p}_x = m\ddot{x} = -\frac{\partial \mathcal{H}}{\partial x} = 0 \qquad \text{so} \qquad \ddot{x} = 0$$

which we know to be correct since there are no forces in the x-direction.
Hamilton's equations for y are

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{p_y}{m} \qquad \text{and} \qquad \dot{p}_y = m\ddot{y} = -\frac{\partial \mathcal{H}}{\partial y} = -mg \qquad \text{so} \qquad \ddot{y} = -g$$

which is also exactly what you expect, that is, the vertical acceleration is just $-g$.