Name: ____________________

PHYS3101 Analytical Mechanics S23 Quiz #10 2 Nov 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A mass m is confined to a long straight rod rotating with fixed angular velocity ω in the horizontal plane. Using the coordinate r measured from the center of the rod,

(a) Construct the Lagrangian $\mathcal{L}(r, \dot{r})$.
(b) Find the conjugate momentum p.
(c) Construct the Hamiltonian $\mathcal{H}(r, p)$.
(d) Determine Hamilton's equations of motion for the mass.
(e) **Extra Credit:** Describe the motion of the mass for $r(0) > 0$ and $p(0) = 0$.

PHY3101 Analytical Mechanics Quiz #10 Solution

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$$\begin{aligned}
\mathcal{L}(r, \dot{r}) &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \\
\mathcal{H}(r, p) &= p\dot{r} - \mathcal{L} = p\frac{p}{m} - \frac{1}{2}m\left(\frac{p}{m}\right)^2 - \frac{1}{2}m\omega^2 r^2 = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 r^2 \\
\dot{r} &= \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \\
\dot{p} &= -\frac{\partial \mathcal{H}}{\partial r} = m\omega^2 r
\end{aligned}$$

Combining Hamilton's equations gives

$$\ddot{r} = \omega^2 r \qquad \text{so} \qquad r(t) = Ae^{\omega t} + Be^{-\omega t}$$

Setting $\dot{p}(0) = m\omega(A - B) = 0$ means $A = B$ so $r(t)$ is proportional to $\cosh(\omega t)$. If $r(0) > 0$ then the coefficient is nonzero and the mass flies off on the rod exponentially. This, of course, agrees perfectly with what you'd expect.