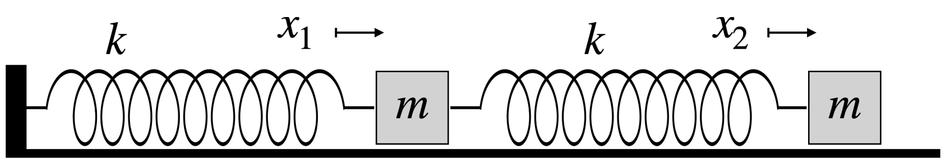
Name: ____________________

PHYS3101 Analytical Mechanics S23 Quiz #9 26 Oct 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Two masses m are connected to each other and to a fixed wall by two identical springs k:



(a) Construct the Langrangian $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2)$.

(b) Find the equations of motion for $\ddot{x}_1$ and $\ddot{x}_2$.

(c) Find the matrices $\underline{\underline{M}}$ and $\underline{\underline{K}}$ to write the equations of motion as $\underline{\underline{M}}\,\underline{\ddot{x}} = -\underline{\underline{K}}\,\underline{x}$.

(d) Find the eigenfrequencies of this system.

PHY3101 Analytical Mechanics Quiz #9 Solution

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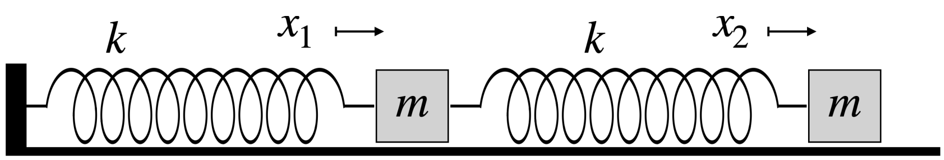

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The Lagrangian is straightforward.

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2$$

The Euler-Lagrange equations give us the equations of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \frac{\partial \mathcal{L}}{\partial x_1} \quad \text{so} \quad m\ddot{x}_1 = -kx_1 - k(x_2 - x_1)(-1) = -2kx_1 + kx_2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{\partial \mathcal{L}}{\partial x_2} \quad \text{so} \quad m\ddot{x}_2 = -k(x_2 - x_1)(-1) = kx_1 - kx_2$$

These can be written as $\underline{\underline{M}}\,\underline{\ddot{x}} = -\underline{\underline{K}}\,\underline{x}$ where

$$\underline{\underline{M}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad \underline{\underline{K}} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

The eigenfrequencies ω^2 are determined from

$$\left|\underline{\underline{K}} - \omega^2\underline{\underline{M}}\right| = \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{vmatrix} = m\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0$$

where $\omega_0^2 \equiv k/m$. The characteristic equation is therefore

$$(2\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

The two solutions for ω^2 are just from the quadratic equation, that is

$$\omega^2 = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2} = \frac{3 \pm \sqrt{5}}{2}\omega_0$$