Name: ________________________________

PHYS3101 Analytical Mechanics S23 Quiz #5 28 Sep 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

We showed in class that the energy of a mass μ in a Kepler orbit is a constant given by

$$E = \frac{1}{2}\mu\dot{r}^2 - \frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

for a central potential $U(r) = -\gamma/r$ and (constant) angular momentum ℓ. By considering the motion of the mass at its points of minimum or maximum approach to the origin (aka the "focus"), derive an expression for E in term of γ, μ, ℓ, and the orbit's eccentricity ϵ. Show that $E < 0$ if $\epsilon < 1$ and $E > 0$ if $\epsilon > 1$.

PHY3101 Analytical Mechanics Quiz #5 <u>Solution</u>

We showed in class that the energy of a mass μ in a Kepler orbit is a constant given by

$$E = \frac{1}{2}\mu\dot{r}^2 - \frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

for a central potential $U(r) = -\gamma/r$ and (constant) angular momentum ℓ. By considering the motion of the mass at its points of minimum or maximum approach to the origin (aka the "focus"), derive an expression for E in term of γ, μ, ℓ, and the orbit's eccentricity ϵ. Show that $E < 0$ if $\epsilon < 1$ and $E > 0$ if $\epsilon > 1$.

See Taylor page 313. The radial speed $\dot{r} = 0$ at the points of either closest or farthest approach. These distances are given by

$$r_{\text{min}} = \frac{c}{1+\epsilon} \qquad \text{and} \qquad r_{\text{max}} = \frac{c}{1-\epsilon} \qquad \text{i.e.} \qquad r_{\text{min,max}} = \frac{c}{1\pm\epsilon} \qquad \text{where} \qquad c = \frac{\ell^2}{\gamma\mu}$$

Plugging these into the equation for the energy (with $\dot{r} = 0$) gives

$$\begin{aligned}
E &= -\frac{\gamma}{c}(1\pm\epsilon) + \frac{\ell^2}{2\mu c^2}(1\pm\epsilon)^2 \\
&= -\frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon) + \frac{\ell^2\gamma^2\mu^2}{2\mu\ell^4}(1\pm\epsilon)^2 \\
&= -\frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon) + \frac{\gamma^2\mu}{2\ell^2}(1\pm\epsilon)^2 \\
&= \frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon)\left[-1 + \frac{1}{2}(1\pm\epsilon)\right] \\
&= \frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon)\left[-1 + \frac{1}{2} \pm \frac{\epsilon}{2}\right] \\
&= \frac{\gamma^2\mu}{\ell^2}(1\pm\epsilon)(-1)\left[\frac{1}{2} \mp \frac{\epsilon}{2}\right] \\
&= -\frac{\gamma^2\mu}{2\ell^2}(1\pm\epsilon)(1\mp\epsilon) = -\frac{\gamma^2\mu}{2\ell^2}(1-\epsilon^2)
\end{aligned}$$

which agrees with Taylor (8.58). This makes it trivial to show that $E < 0$ if $\epsilon < 1$ and $E > 0$ if $\epsilon > 1$, consistent with Taylor Fig.8.4 for elliptical ($\epsilon < 1$) and hyperbolic ($\epsilon > 1$) orbits.