Name: ______________________________

# PHYS3101 Analytical Mechanics S23 Quiz #4 21 Sep 2023

*You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.*

**Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.**

A mass $m$ is tied by a massless string to a point on a horizontal, frictionless table. The mass is executing circular motion with radius $R$ at constant speed $v = R\dot{\phi}$. Assume there are perpendicular $x$ and $y$ axes with origin at the fixed point of the string. Use the method of Lagrange multipliers to find the constraint forces, and show that this is what you expect by equating the string tension to the centripetal acceleration of the mass.

# PHY3101 Analytical Mechanics Quiz #4 Solution

A mass $m$ is tied by a massless string to a point on a horizontal, frictionless table. The mass is executing circular motion with radius $R$ at constant speed $v = R\dot{\phi}$. Assume there are perpendicular $x$ and $y$ axes with origin at the fixed point of the string. Use the method of Lagrange multipliers to find the constraint forces, and show that this is what you expect by equating the string tension to the centripetal acceleration of the mass.

Write the constraint as $f(x, y) = x^2 + y^2 = R^2$. There is no potential energy term, so

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

The modified Lagrange equations are then

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 2\lambda x - m\ddot{x} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial y} + \lambda\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 2\lambda y - m\ddot{y} = 0$$

The constraints are realized with the parameterization $x = R\cos\phi$ and $y = R\sin\phi$ so

$$\begin{array}{rlcl} & \dot{x} = -R\dot{\phi}\sin\phi & \text{so} & \ddot{x} = -R\dot{\phi}^2\cos\phi - R\ddot{\phi}\sin\phi \\ \text{and} & \dot{y} = R\dot{\phi}\cos\phi & \text{so} & \ddot{y} = -R\dot{\phi}^2\sin\phi + R\ddot{\phi}\cos\phi \end{array}$$

The two modified Lagrange equations become

$$\begin{array}{rrcl} & 2\lambda R\cos\phi + mR\dot{\phi}^2\cos\phi + mR\ddot{\phi}\sin\theta & = & 0 \\ \text{and} & 2\lambda R\sin\phi + mR\dot{\phi}^2\sin\phi - mR\ddot{\phi}\cos\phi & = & 0 \end{array}$$

Multiply the first equation by $\cos\phi$ and the second by $\sin\phi$ and add to get

$$2\lambda = -m\dot{\phi}^2$$

The constraint forces are therefore

$$F_x = \lambda\frac{\partial f}{\partial x} = 2\lambda x = -m\dot{\phi}^2 R\cos\phi \qquad \text{and} \qquad F_y = \lambda\frac{\partial f}{\partial y} = 2\lambda y = -m\dot{\phi}^2 R\sin\phi$$

From centripetal acceleration considerations, the tension (which is negative in both $x$ and $y$ components in the first quadrant) is given by

$$T = m\frac{v^2}{R} = mR\dot{\phi}^2$$

so the analysis above indeed gives the correct $x$ and $y$ components.