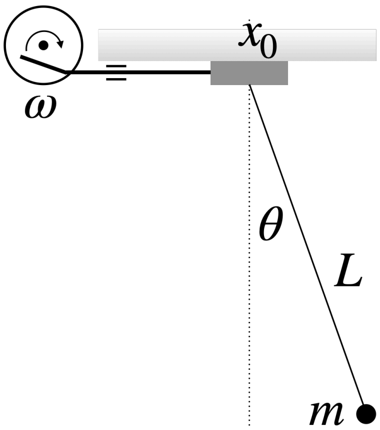
Name: ______________________________

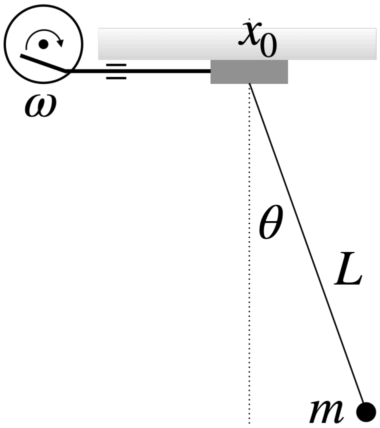
PHYS3101 Analytical Mechanics S23 Quiz #3 14 Sep 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.



A pendulum made from a string of length L and bob mass m swings in the vertical plane through an angle θ. It hangs under gravity near the Earth's surface. The top of the string is fixed vertically but moves horizontally with a position $x_0(t) = A \sin \omega t$. Find the (differential) equation of motion for the angle $\theta(t)$. Show that the answer is what you expect if $\omega = 0$.



A pendulum made from a string of length L and bob mass m swings in the vertical plane through an angle θ. It hangs under gravity near the Earth's surface. The top of the string is fixed vertically but moves horizontally with a position $x_0(t) = A\sin\omega t$. Find the (differential) equation of motion for the angle $\theta(t)$. Show that the answer is what you expect if $\omega = 0$.

This problem is very similar to the homework problem with the elevator accelerating upward, but now the pivot point of the pendulum is moving sideways instead of upward, and you are explicitly given its motion.

The position of the mass, with $y = 0$ at the bottom of the pendulum swing, is given by

$$\begin{aligned} x &= x_0 + L\sin\theta = A\sin\omega t + L\sin\theta \\ y &= L(1-\cos\theta) \end{aligned}$$

The velocities are

$$\begin{aligned} \dot{x} &= \omega A\cos\omega t + L\dot{\theta}\cos\theta \\ \dot{y} &= L\dot{\theta}\sin\theta \end{aligned}$$

The Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy \\ &= \frac{1}{2}m(L^2\dot{\theta}^2 + \omega^2A^2\cos^2\omega t + 2\omega AL\dot{\theta}\cos\omega t\cos\theta) + mgL\cos\theta + \text{constant} \end{aligned}$$

The equation of motion is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} &= \frac{d}{dt}\left[mL^2\dot{\theta} + m\omega AL\cos\omega t\cos\theta\right] + m\omega AL\dot{\theta}\cos\omega t\sin\theta + mgL\sin\theta \\ &= mL^2\ddot{\theta} - m\omega^2AL\sin\omega t\cos\theta - m\omega AL\dot{\theta}\cos\omega t\sin\theta \\ &\quad + m\omega AL\dot{\theta}\cos\omega t\sin\theta + mgL\sin\theta \\ &= mL^2\ddot{\theta} - m\omega^2AL\sin\omega t\cos\theta + mgL\sin\theta = 0 \end{aligned}$$

This simplifies to

$$\ddot{\theta} + \omega_0^2\sin\theta = \frac{A}{L}\omega^2\sin\omega t\cos\theta \qquad \text{where} \qquad \omega_0^2 \equiv \frac{g}{L}$$

is the natural frequency of the simple pendulum. For $\omega = 0$ this is just the equation of motion for a simple pendulum, as expected. Notice that the result simplifies to a forced, undamped harmonic oscillator for $\theta \ll 1$.