Name: ____________________

PHYS3101 Analytical Mechanics S23 Quiz #2 7 Sep 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

An object of mass m is attracted to the origin by a force $F = -k/r^2$ where k is a constant, and r is the distance to the origin. Let r and ϕ be the usual polar coordinates in a plane.

(a) Find the potential energy function $U(r)$ where $U(r \to \infty) = 0$.

(b) Construct the Lagrangian $\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t)$.

(c) Show that the quantity $\ell \equiv mr^2\dot{\phi}$ is a constant in time.

(d) Find the differential equation of motion for r in terms of m, k, and ℓ.

PHY3101 Analytical Mechanics Quiz #2 Solution

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(d) Find the differential equation of motion for r in terms of m, k, and ℓ.

The potential energy function is a familiar result. Using (4.13) in Taylor, with $r_0 = \infty$,

$$U(r) = -\int_\infty^r F(r')\,dr' = \int_\infty^r \frac{k}{r'^2}\,dr' = \left[-\frac{k}{r'}\right]_\infty^r = -\frac{k}{r}$$

For the Lagrangian, use the kinetic energy $m\mathbf{v}^2/2$ in polar coordinates (1.43) in Taylor so

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} - U(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{k}{r}$$

The Lagrange equation for ϕ is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = 0 \qquad \text{so} \qquad mr^2\dot{\phi} = \text{constant} \equiv \ell$$

The Lagrange equation for r is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\partial\mathcal{L}}{\partial r} \qquad \text{so} \qquad m\ddot{r} = mr\dot{\phi}^2 - \frac{k}{r^2} = \frac{\ell^2}{mr^3} - \frac{k}{r^2}$$