PHYS3101 Analytical Mechanics Homework #12 Due 28 Nov 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A taut string has a fixed end at $x = -a$ and extends infinitely to positive x. The string is initially at rest and has the shape of an isosceles triangular pulse centered on $x = 0$ and extending over $-a/5 \leq x \leq a/5$. Find the equation for the shape of the string at all times, and plot it for $t = 0$, $t = 0.5a/c$, $t = a/c$, and $t = 1.5a/c$ where c is the speed of the wave on a string. If you prefer, you can create an animation. (Note that in MATHEMATICA, you can easily define this kind of function using HeavisideLambda.)

(2) In class we derived expressions for the kinetic and potential energies of a taut string as integrals over the length of the string. Using what you have previously learned for normal modes of the shape $u(x, t)$ of a stretched string of length L fixed to $u = 0$ at $x = 0$ and $x = L$, see Concepts (5.12), write the total energy of the string as a single sum over the normal modes and show that it is a constant in time.

(3) The wave equation in three spatial dimensions is

$$\boldsymbol{\nabla}^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

If $f(\mathbf{r}, t) = f(r, t)$ where r is the usual spherical polar coordinate, show that

$$f(r, t) = \frac{A}{r} e^{i(kr - \omega t)}$$

solves the wave equation, where A is a constant and $\omega = ck$.

(4) The equation of motion for an inviscid fluid of density $\rho(\mathbf{r}, t)$ in a gravitational field $\mathbf{g}$ is

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} p$$

where $\mathbf{v}(\mathbf{r}, t)$ is the velocity field and $p(\mathbf{r}, t)$ is the pressure field. Use this to show the familiar result from your first physics course that the difference in pressure between two points in a static and incompressible fluid separated by a vertical distance h is $\Delta p = \rho g h$.

(5) Find the speed of sound in air, using the following steps, and compare with the accepted value of 331 m/s at 0°C. First show that the bulk modulus of air is γp, where p is pressure and $\gamma \equiv C_p/C_V = 1.4$ is the ratio of specific heats for an ideal gas. You can assume that the adiabatic expansion and compression in air, as a sound wave passes, follows $pV^\gamma =$ constant, where V is the volume. Then use the ideal gas law $pV = NkT$ to express density of N air molecules with mass m in terms of p, m, and T. Finally, combine these two results using the formalism we developed in class. Repeat the calculation for helium gas instead of nitrogen, and explain why your voice sounds high pitched if you first breath in some helium before speaking.