PHYS3101 Analytical Mechanics   Homework #10   Due 7 Nov 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Consider any two functions of generalized coordinates and momenta $f(\underline{q}, \underline{p})$ and $g(\underline{q}, \underline{p})$. Show each of the following for the Poisson Bracket, which is defined as

$$[f, g] \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

(a) $[g, f] = -[f, g]$
(b) $[q_i, q_j] = 0 = [p_i, p_j]$ but $[q_i, p_j] = \delta_{ij}$
(c) $\dot{q}_i = [q_i, \mathcal{H}]$ and $\dot{p}_i = [p_i, \mathcal{H}]$

If you've studied some Quantum Mechanics, does any of this look familiar to you?

(2) Write down the Hamiltonian for a simple plane pendulum of length ℓ with a bob of mass m. Use ϕ for the angle of the bob measured with respect to the vertical, as usual with $\phi = 0$ for the mass at its lowest point, and define $\omega^2 = g/\ell$.

(a) Find Hamilton's equations for ϕ and its conjugate momentum p. These equations become particularly simple of you write them in terms of dimensionless $\tilde{p} = p/m\ell^2\omega$.
(b) Plot $\phi(t)$, $p(t)$, and also the orbit (p versus ϕ) for the three sets of initial conditions (i) $\phi(0) = 0.1$ and $p(0) = 0$; (ii) $\phi(0) = 0.99\pi$ and $p(0) = 0$; and (iii) $\phi(0) = 0.99\pi$ and $p(0) = -0.05$. Briefly explain the motion these each describe.

(3) A beam of particles is moving along an accelerator pipe in the z-direction. The particles are uniformly distributed in a cylindrical volume of length L_0 (in the z-direction) and radius R_0. The particles have momenta uniformly distributed with p_z in an interval $p_0 \pm \Delta p$ and the transverse momentum $p_\perp$ inside a circle of radius $\Delta p_\perp$. To increase the particles' spatial density, the beam is focused by electric and magnetic fields, so that the radius shrinks to a smaller value R. What does Liouville's theorem tell you about the spread in the transverse momentum $p_\perp$ and the subsequent behavior of the radius R? (Assume that the focusing does not affect either L_0 or Δp_z.) Google "stochastic cooling" to learn why this is important.

(4) Suppose that you believe that antiprotons $\bar{p}$ exist, and you want to build an accelerator that would be able to produce them using the reaction $p + p \to p + p + p + \bar{p}$ where an incident beam proton of kinetic energy T is incident on a stationary target proton. What is the minimum amount of energy you need to produce antiprotons in this reaction? Can you find the name and location of the accelerator that was built to carry out this experiment?

(5) A particle of mass m has initial kinetic energy $T \gg mc^2$ and scatters from a stationary, identical particle. If the scattering is at 90° in the center of mass frame, find the outgoing opening angle between the two particles in the laboratory frame. You can work this out easily enough just by using conservation of energy and momentum, but a slick solution is to equate two inner products in the lab and CM frames. Be careful of how you use $T \gg mc^2$.