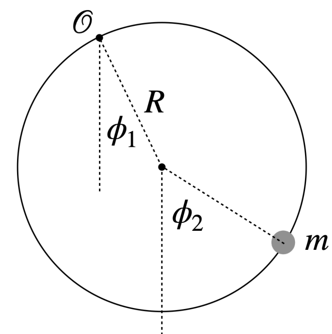
PHYS3101 Analytical Mechanics Homework #8 Due 24 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) If $\underline{\underline{I}}^{\text{CM}}$ is the moment of inertia tensor for some rigid body of mass M about its center of mass, and $\underline{\underline{I}}$ is the inertia tensor about a point displaced an amount $\mathbf{\Delta} = \hat{\mathbf{x}}\Delta_x + \hat{\mathbf{y}}\Delta_y + \hat{\mathbf{z}}\Delta_z$ from the center of mass, show that

$$\underline{\underline{I}}_{ij} = \underline{\underline{I}}_{ij}^{\text{CM}} + M\left(\mathbf{\Delta}^2\delta_{ij} - \Delta_i\Delta_j\right)$$

(2) A hoop of mass m and radius R hanging from a fixed point $\mathcal{O}$ swings freely in the vertical plane. A bead, also of mass m, slides without friction on the hoop. Using the angles ϕ_1 and ϕ_2 as shown in the figure on the right as generalized coordinates, write the Lagrangian. (You will need to find the moment of inertia of the hoop about a point on the hoop. This is a very simple calculation if you use the results of Problem **(1)**.) Then for $\phi_1 \ll 1$ and $\phi_2 \ll 1$, determine the equations of motion and find the eigenfrequencies and describe the normal modes.



(3) Two equal masses m are connected by three springs and slide freely on a horizontal track. The outer springs are attached to fixed walls and have stiffness k, and the middle spring has stiffness εk. (Note that this ε is dimensionless, and is not the same as ϵ used in Taylor Section 11.3.) Derive expressions for the positions $x_1(t)$ and $x_2(t)$ of the two masses assuming they both start from rest and $x_1(0) = a$ and $x_2(0) = 0$. (You can carry out this calculation any way you like, including just solving the differential equations using MATHEMATICA.) Make plots of $x_1(t)/a$ and $x_2(t)/a$ for $\epsilon = 1/10$ as a function of $\tau = \Omega t$ where $\Omega^2 = k/m$. Your result should look like Figure 11.8 in Taylor. Also plot the normal modes and show that they oscillate with two distinct, single frequencies.

(4) Consider a potential energy function $U(q_1, q_2, \ldots, q_n)$ where the q_i are n generalized coordinates that describe a system of N masses. Assume that when all of the $q_i = 0$, then the function U is at a local minimum. Find the Euler-Lagrange equations of motion for the case when the q_i do not move far from equilibrium, and show that the equations of motion can be written as

$$\underline{\underline{M}}\,\underline{\ddot{q}} + \underline{\underline{K}}\,\underline{q} = 0$$

where $\underline{\underline{M}}$ and $\underline{\underline{K}}$ are real, symmetric matrices. (This is most easily done by deriving expressions for $\underline{\underline{M}}$ and $\underline{\underline{K}}$ in terms of what you are given. See Taylor Section 7.8.)

(5) Consider a frictionless rigid horizontal hoop of radius R. Onto this hoop are threaded three beads with masses $2m$, m, and m, and, between the beads, three identical springs, each with force constant k. Solve for the three normal frequencies and find and describe the three normal modes.