PHYS3101 Analytical Mechanics Homework #7 Due 17 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Prove that the principle moments of inertial λ_1, λ_2, and λ_3 of any rigid body must satisfy $\lambda_3 \leq \lambda_1 + \lambda_2$. If $\lambda_3 = \lambda_1 + \lambda_2$, what does that imply about the shape of the body? Pick a specific example to check your answer.

(2) A thin, flat, uniform rectangular plate with mass M lies in the xy plane with two of its corners at $(a, b, 0)$ and the origin. Find the plate's inertia tensor and then diagonalize it to find the principle moments of inertia and the principle axes. Comment on the comparison between $\lambda_1 + \lambda_2$ and λ_3, based on Problem **(1)** above. Choose values for a and b and draw diagrams that show the principle axes for the plate if $a = b$ (a square) and $a \neq b$.

(3) For a symmetric rigid body rotating in free space with no external torques, we showed in class that Euler's Equations implied that, in the body frame (for example, the Earth in Problem 5 of Homework #6) both the angular velocity vector $\boldsymbol{\omega}$ and the angular momentum vector $\mathbf{L}$ precessed around the symmetry axis $\hat{\mathbf{e}}_3$ with a frequency $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1$. Now find the the space frame frequency Ω_s at which $\boldsymbol{\omega}$ and $\hat{\mathbf{e}}_3$ precess about $\mathbf{L}$ is $\Omega_s = L/\lambda_1$. See Figure 10.9 in Taylor. You can do this by first explaining why $\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_b + \boldsymbol{\omega}$. Then consider the angles between $\hat{\mathbf{e}}_3$ and $\boldsymbol{\omega}$, and between $\hat{\mathbf{e}}_3$ and $\mathbf{L}$.

(4)A symmetric top of mass M spins about its symmetry axis at an angular speed ω_3, with a fixed point at the origin. The distance from the origin to the center of mass is R. If the top precesses at a fixed angle θ, show that

$$\lambda_1 \Omega^2 \cos\theta - \lambda_3 \omega_3 \Omega + MgR = 0$$

where λ_1 and λ_3 are principle moments of inertia, and Ω is the rate of precession. Assuming that ω_3 is "very large", solve this quadratic equation for the two possible values of Ω. What kind of motions do these two solutions represent? What does "very large" mean for ω_3? That is, very large compared to what?

(5) The effective potential energy for a spinning symmetric top is

$$U_{\text{eff}}(\theta) = \frac{(L_z - L_3 \cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

where θ is the polar angle from the vertical, L_z and L_3 are the angular momenta vertical and symmetry body axis, respectively, and λ_1 and λ_3 are principle moments of inertia, and M is the mass. The fixed point is at the origin, and R is the distance from the origin to the center of mass. Why is the second term unimportant for understanding the motion of the top? Plot $U_{\text{eff}}(\theta)$ for $\lambda_1 = 1 = MgR$, $L_z = 8$, and $L_3 = 10$, and find to three significant figures the value θ_0 at which the top precesses with constant θ. Find the rate $\Omega \equiv \dot{\phi}$ of steady precession from the equation for L_z, and compare to the approximate result you obtain in the case where the top is spinning "very rapidly"; see Problem **(4)** above.