PHYS3101 Analytical Mechanics Homework #5 Due 3 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A mass m follows an elliptical orbit for a potential energy $U(r) = -\gamma/r$. Show that

$$\boldsymbol{\epsilon} \equiv \frac{1}{\gamma}\dot{\mathbf{r}} \times \boldsymbol{\ell} - \hat{\mathbf{r}}$$

where $\boldsymbol{\ell} = \mathbf{r} \times m\dot{\mathbf{r}}$, is a (vector) constant of the motion, and that its magnitude is the eccentricity of the ellipse. What physical characteristic of the ellipse is described by the *direction* of $\boldsymbol{\epsilon}$? *Hints and comments:* Make use of vector identities with cross products, including $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$, and $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. To find $\epsilon = |\boldsymbol{\epsilon}|$ do $\epsilon^2 = \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}$, assemble terms, and recognize the total energy. For the direction, use a simple dot product to show that $\boldsymbol{\epsilon}$ lies in the plane of the orbit. Then, evaluate $\boldsymbol{\epsilon} \cdot \mathbf{r}$ assuming that the angle between $\boldsymbol{\epsilon}$ and $\mathbf{r}$ is α. Solve for r, and compare to the formula for an ellipse to discover the relationship between α and ϕ. (Your instincts could tell you the direction of $\boldsymbol{\epsilon}$; the "obvious" choice turns out to be correct.)

(2) A 120 lb person sits in a Chevy Corvette, which accelerates from zero to 60 miles per hour in six seconds. (That's faster than most cars.) Assuming the acceleration is constant, how much force (in pounds) does she feel pushing her against the back of the seat? A weight hangs from a string suspended from the ceiling. What angle does it make with the vertical?

(3) An observer sits on a turntable which rotates counter clockwise at a constant angular speed Ω. A mass m rides on the frictionless surface of the turntable. The observer sees the mass move in a circle of radius R at fixed angular velocity ω about the same axis as the turntable. Find the value of ω such that the combined centrifugal and Coriolis forces give just the right centripetal force $m\omega^2 R$, directed towards the center of rotation, to maintain the circular motion. What's going on? (The answer should be obvious.)

(4) On a certain planet, which is perfectly spherically symmetric, the free fall acceleration has magnitude $g = g_0$ at the North Pole and $g = \lambda g_0$ at the equator (with $0 \leq \lambda \leq 1$). Find $g(\theta)$, the free fall acceleration at a colatitude θ as a function of θ.

(5) Use the method of successive approximations to find the path $\mathbf{r}(t)$, to first order in the Earth's rotation speed Ω, of an object thrown from an origin located at colatitude θ with initial velocity $\mathbf{v}_0 = v_{x_0}\hat{\mathbf{x}} + v_{y_0}\hat{\mathbf{y}} + v_{z_0}\hat{\mathbf{z}}$. Assume that the acceleration vector $\mathbf{g}$ due to gravity is constant throughout the flight, and ignore air resistance.