PHYS3101 Analytical Mechanics Homework #4 Due 26 Sep 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) We showed that the Lagrangian for a system of two masses m_1 and m_2, which only interact through a "central" potential $U(|\mathbf{r}_1 - \mathbf{r}_2|)$, decouples into a center-of-mass (CM) coordinate $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$ where $M = m_1 + m_2$, and a relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

(a) Show that the potential energy $U(\mathbf{r}) = -F\mathbf{r}\cdot\hat{\mathbf{n}}$ for a uniform force field $\mathbf{F} = F\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is an arbitrary unit vector.

(b) Show that even in the presence of an external force $\mathbf{F} = ma\hat{\mathbf{n}}$, where a is a constant and m is the mass, the Lagrangian still decouples into CM and relative coordinates.

(c) Explain why the Earth-Moon system can be described (to a very good approximation) as a two-body central force problem, even though both are orbiting about the Sun.

(2) Two bodies with masses m and M orbit each other based on their mutual gravitational attraction. If we don't make the assumption that $m \ll M$ for Keplerian orbits, show that the correct form of Kepler's Third Law is

$$\tau^2 = \frac{4\pi^2}{G(M+m)}a^3$$

for period τ and semimajor axis a. Use an elementary "$F = ma$" approach, where a is the centripetal acceleration, to show that this is correct for two stars of the same mass following circular orbits about their common center of mass.

(3) Calculate the radius of a stable circular orbit by finding the minimum in the effective potential energy for a mass m orbiting a much larger mass M. Use Use an elementary "$F = ma$" approach, where a is the centripetal acceleration, to show that you got the correct answer. Then, calculate the period of small oscillations about this point by expanding the effective potential in a Taylor series, and compare to the period of the circular orbit.

(4) In General Relativity, Newtonian gravity is modified so that the force is

$$F(r) = -G\frac{mM}{r^2}\left(1 - \frac{R_s}{r}\right)$$

where $R_s = 2GM/c^2$ is the "Schwarzschild radius" of the large mass M, and is very much smaller than the orbital distance r for planets in our Solar System. Show that the orbit is very nearly an ellipse (for negative total energy), and calculate by how much the axis of the ellipse precesses (in angle) over one orbit, if the orbit is nearly circular with radius R.

(5) Calculate the amount of time (in years) it would take to launch a spacecraft from Earth to Uranus, using the most direct path possible, using no fuel other than to shoot it out of Earth's orbit. Assume that Earth and Uranus are in circular orbits with radii 1 AU and 19.2 AU. You can also assume that the launch happens at the right time so that Uranus is in the right place when the spacecraft arrives at its orbit.