PHYS3101 Analytical Mechanics Homework #1 Due 5 Sep 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A man of mass M straddles the gap between a train and the station platform, with his left foot on the train step and his right on the platform. He lifts his right foot at the same time as he jerks his bag with mass m onto the train. He pulls the bag with a constant horizontal acceleration and it lands on the train in a time T. If the distance between the bag and the train is L and the coefficient of static friction between his left foot and the train step is μ, derive a condition that can be tested to see if his foot slips and he falls into the gap. Use the average horizontal velocity of the bag to calculate the impulse on the man. If $M = 110$ kg, $L = 1$ m, $T = 1/4$ s, $m = 20$ kg, and $\mu = 0.4$, does the man fall into the gap?

(2) The force $\mathbf{F}$ on a particle with mass m and charge q moving with velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is given by $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. If $\mathbf{E} = E\hat{\mathbf{z}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, solve for the motion $\mathbf{r}(t)$ if the initial conditions are that the particle starts at the origin and

(a) $\mathbf{v}(t = 0) = 0$
(b) $\dot{x} = v_0$, and $\dot{y} = \dot{z} = 0$

(3) Use MATHEMATICA or some other application to reproduce the plot at the bottom of Figure 5.15 from Example 5.3 in the textbook. You are welcome to arrive at the solution with whatever tools you'd like. (I just used DSolve.) This example uses a damping coefficient $\beta = \omega_0/20$. Now repeat the calculation for $\beta = \omega_0/5$ and for $\beta = \omega_0/50$. Explain the physical origin of the differences and similarities of these three plots.

(4) Find an equation for the path $y = f(x)$ that makes the integral

$$S = \int_a^b x \left[1 - (f'(x))^2\right]^{1/2} dx$$

an extremum for fixed values of a and b.

(5) Consider a mass m moving in two dimensions (x, y) or (r, ϕ), subject to a potential energy function $U = kr^2/2$ where k is a constant. Write down the Lagrangian, find the equations of motion, and describe their solutions, when the particle's position is written in terms of

(a) coordinates x and y.
(b) coordinates r and ϕ.

For part (b), what is the significance of the solution for $\phi(t)$?

PHYS3101 Analytical Mechanics Homework #1 Solutions

(1) The acceleration a of the bag is given by $L = aT^2/2$, so $a = 2L/T^2$. The velocity increases linearly with time, so the average velocity is $v = aT/2 = L/T$. (Of course.) The momentum of the bag is $mv = mL/T$ so the average backward impulse on the man is $FT = mL/T$ where $F = mL/T^2$ is the force pulling the man backward. If this overcomes the frictional force μMg, the man's food will slip and he'll fall through the gap. That is, if

$$mL/T^2 > \mu Mg$$

then the man will fall. For the given numbers, $mL/T^2 = 20(1)/(1/4)^2 = 320$ N, and $\mu Mg = 0.4(110)9.8 = 431$ N, so he shouldn't fall. But it's close.

(2) See Section 2.5 in Taylor. Modified versions of (2.64), (2.65), and (2.66) give us

$$m\dot{v}_x = qBv_y \qquad m\dot{v}_y = -qBv_x \qquad m\dot{v}_z = qE$$

The z-equation is decoupled from the other two, so for both (a) and (b), the solution is $v_z = qEt/m$ and $z = qEt^2/2m$. From (2.71), $v_x = A\cos(\omega t) = v_0\cos(\omega t)$ where $\omega = qB/m$, and $v_y = -A\sin(\omega t) = -v_0\sin(\omega t)$ which satisfy the initial conditions on the velocity. For (a), $v_0 = 0$ and the charge just accelerates in a straight line along the z-axis. For (b), integrate to get $x = (v_0/\omega)\sin(\omega t)$ and $y = (v_0/\omega)[\cos(\omega t) - 1]$ satisfying the initial conditions on position.

(3) See the accompanying MATHEMATICA notebook.

(4) Writing $S = \int_a^b F(f, f', x)\, dx$ (the notation used in Concepts) we apply the Euler-Lagrange equation to the function $F = x\left[1 - (f'(x))^2\right]^{1/2}$. Since F does not explicitly depend on f, this means that $\partial F/\partial f' = \text{constant} \equiv a$. Therefore

$$-\frac{xf'(x)}{\left[1 - (f'(x))^2\right]^{1/2}} = a \qquad \text{so} \qquad f'(x) = \frac{dy}{dx} = \frac{a}{\sqrt{x^2 + a^2}}$$

This is simple to integrate using $x = a\sinh t$ since $x^2 + a^2 = a^2\cosh^2 t$ and $dx = a\cosh t$ so $y = b + t = b + \sinh^{-1}(x/a)$. See Problem 6.12 in Taylor.

(5) We know that $r^2 = x^2 + y^2$ and $\mathbf{v}^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2$ so the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k\left(x^2 + y^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{1}{2}kr^2$$

In Cartesian coordinates, this gives $m\ddot{x} + kx = 0$ and $m\ddot{y} + ky = 0$, so the motion is two independent simple harmonic oscillators with frequency $\omega = \sqrt{k/m}$. In polar coordinates, the ϕ equation gives $mr^2\dot{\phi} = \text{constant} \equiv \ell$, the angular momentum. Then

$$mr\dot{\phi}^2 - kr - m\ddot{r} = 0 \qquad \text{or} \qquad m\ddot{r} = -kr + \frac{\ell^2}{2mr^2}$$

which is a radial simple harmonic oscillation with a "centrifugal" force term. (More on that when we study non-inertial reference frames.)

PHYS3101 Analytical Mechanics Homework #2 Due 12 Sep 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A simple plane pendulum of length ℓ hangs from the ceiling of an elevator. The elevator is accelerating upwards at an acceleration a. (If $a < 0$, then the elevator is accelerating downwards.) Use the Lagrangian approach to find the frequency of small oscillations in terms of ℓ, a, and g. (This problem is a demonstration of Einstein's Principle of Equivalence.) It is probably easiest to write the Lagrangian first in terms of Cartesian coordinates.

(2) A mass M hang from a massless string that passes over a massless pulley. A similar pulley hangs from the other end of the string, over which a second massless string supports a mass m_1 on one side and a mass m_2 on the other side. Use Lagrange's equations to find the accelerations of M and m_1 when the system is released, in terms of m_1, m_2, M, and g. (If you find the necessary algebra at the end a little messy, I suggest you do it with MATHEMATICA.) Under what conditions is the acceleration of M equal to zero?

(3) A helix is determined in terms of its radius R and pitch λ using three dimensional spherical coordinates ρ, ϕ, and z as $\rho = R$ and $z = \lambda\phi$. If a particle of mass m is constrained to lie on the helix, which is oriented so that z is vertical, use Lagrange's equations to find the acceleration $\ddot{z}$ in terms of R, λ, and g. Discuss the behavior of $\ddot{z}$ for limiting values of R and λ, that is the cases $R \ll \lambda$ and $R \gg \lambda$.

(4) A particle of mass m is constrained to move without friction along a horizontal circular hoop of radius R. The hoop rotates with a fixed angular velocity ω about a fixed point on the circle. Show that the motion of the mass is the same as that for a vertical, plane pendulum, and find the frequency of small oscillations. Check that your answer is dimensionally correct. (In fact, if you think about the concept of a "centrifugal force", then you can check that you got the correct answer very simply.)

(5) Example 7.6 in Taylor derives the equation of motion (7.69) for the "bead on a spinning hoop" problem. Rewrite this equation in terms of dimensionless time $x \equiv \omega t$ and $\alpha \equiv g/\omega^2 R$. Assuming the bead starts from rest, numerically solve for $\theta(x)$ and plot it for several periods for each of the following cases:

(a) $\theta_0 = 0.1$ and $\alpha = 2$
(b) $\theta_0 = 1.1$ and $\alpha = 1/2$
(c) $\theta_0 = 0.1$ and $\alpha = 1/2$

Discuss the motion in each case, and compare to what you get from the linearized versions (7.72) or (7.80) in Taylor.

(1) Let x and y locate the horizontal and vertical position of the pendulum bob. Then

$$x = \ell \sin\phi \qquad \text{and} \qquad y = \ell(1-\cos\phi) + \frac{1}{2}at^2$$

So

$$\dot{x} = \ell\dot{\phi}\cos\phi \qquad \text{and} \qquad \dot{y} = \ell\dot{\phi}\sin\phi + at$$

The Lagrangian is therefore

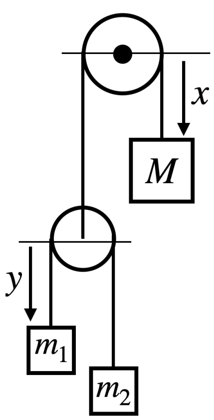
$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy = \frac{1}{2}m(\ell^2\dot{\phi}^2\cos^2\phi + \ell^2\dot{\phi}^2\sin^2\phi + 2\ell\dot{\phi}at\sin\phi + a^2t^2) - mgy \\ &= \frac{1}{2}m\ell^2\dot{\phi}^2 + m\ell\dot{\phi}at\sin\phi + a^2t^2 - mg\ell(1-\cos\phi) - \frac{1}{2}mgat^2\end{aligned}$$

Lagrange's equation for ϕ is therefore

$$\frac{d}{dt}\left(m\ell^2\dot{\phi} + m\ell at\sin\phi\right) = m\ell^2\ddot{\phi} + m\ell a\sin\phi + m\ell at\dot{\phi}\cos\phi = m\ell\dot{\phi}at\cos\phi - mg\ell\sin\phi$$

which reduces to $\ddot{\phi} = -(g+a)/\ell\sin\phi$. This is just the equation for a simple pendulum but with g replaced by $g+a$. That is, the acceleration due to gravity is changed by the acceleration from the elevator, so an upward acceleration appears simply as an increase in gravity. If the elevator is in free fall, $a=-g$ and there pendulum appears to be "weightless." The frequency of small oscillations is just $\omega = \sqrt{(g+a)/\ell}$.

(2) The first step is draw a clear figure, and pick coordinates for the two degrees of freedom.



Let x measure the downward position of M relative to the fixed pulley, and y for m_1 relative to the moving pulley. If L is the length of the strings, then the downward distances of m_1 and m_2 relative to the fixed pulley are $q_1 = L - x + y = L - (x-y)$ and $q_2 = L - x + (L-y) = 2L - (x+y)$. Therefore, the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + Mgx + m_1gq_1 + m_2gq_2 \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1(\dot{x}-\dot{y})^2 + \frac{1}{2}m_2(\dot{x}+\dot{y})^2 \\ &\quad + Mgx - m_1g(x-y) - m_2g(x+y) + \text{constant}\end{aligned}$$

The Lagrange equations are

$$\begin{aligned}M\ddot{x} + m_1(\ddot{x}-\ddot{y}) + m_2(\ddot{x}+\ddot{y}) &= Mg - (m_1+m_2)g \\ -m_1(\ddot{x}-\ddot{y}) + m_2(\ddot{x}+\ddot{y}) &= (m_1-m_2)g\end{aligned}$$

Solve these with Mathematica (see the accompanying notebook) to get

$$\ddot{x} = -\frac{m_1(M-4m_2)+m_2M}{m_2M + m_1(4m_2+M)}g \qquad \text{and} \qquad \ddot{y} = \frac{2(m_1-m_2)M}{m_2M+m_1(4m_2+M)}g$$

Note that this gives $\ddot{x} = g/7$, that is downward, when $M = 4m$, $m_1 = 3m$, and $m_2 = m$, in agreement with the solution manual for Problem 7.27 in Taylor.

Now it kind of makes sense that there should be no acceleration if $m_1 = m_2$ and $M = 2m_1$, but let's confirm it. Writing $m \equiv m_1$, the numerator of $\ddot{x}$ becomes

$$m(2m - 4m) + m(2m) = -2m^2 + 2m^2 = 0$$

and indeed $\ddot{x} = 0$. (Note that this solution also gives $\ddot{y} = 0$.) To find other solutions for $\ddot{x} = 0$, write $m_1 = m$, $m_2 = xm$, and $M = ym$. Then we need $y - 4x + xy = 0$, so

$$y = \frac{4x}{1+x}$$

and the acceleration of M is zero, although in general the acceleration of the m_1 is nonzero.

(3) The Lagrangian is

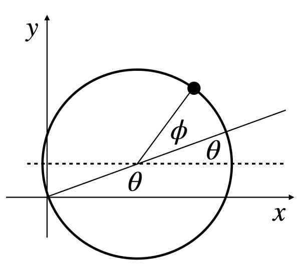
$$\mathcal{L} = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 - mgz = \frac{1}{2}mR^2\left(\frac{\dot{z}}{\lambda}\right)^2 + \frac{1}{2}m\dot{z}^2 - mgz$$

The equation of motion is therefore

$$m\left[\frac{R^2}{\lambda^2} + 1\right]\ddot{z} = -mg \qquad \text{so} \qquad \ddot{z} = -\frac{\lambda^2}{R^2 + \lambda^2}g$$

which agrees with the solution to Problem 7.20 in Taylor. For $R \ll \lambda$, you get $\ddot{z} = -g$ which is what you expect for a very skinny helix. For $R \gg \lambda$, $\ddot{z} \to 0$, which is what you expect when the helix becomes, essentially, a flat circle.

(4) The first step is draw a clear figure.



The radius of the circle is R and the angle $\theta = \omega t$. The (x, y) coordinates of the center of the circle are $(R\cos\theta, R\sin\theta)$, so the x and y coordinates of the mass are

$$\begin{aligned} x &= R\cos\theta + R\cos(\theta + \phi) = R\cos\omega t + R\cos(\omega t + \phi) \\ y &= R\sin\theta + R\sin(\theta + \phi) = R\sin\omega t + R\sin(\omega t + \phi) \end{aligned}$$

There is no potential energy, so the Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\ &= \frac{1}{2}mR^2\left\{\left[-\omega\sin\omega t - (\omega + \dot{\phi})\sin(\omega t + \phi)\right]^2 + \left[\omega\cos\omega t + (\omega + \dot{\phi})\cos(\omega t + \phi)\right]^2\right\} \\ &= \frac{1}{2}mR^2\left\{\omega^2 + 2\omega(\omega + \dot{\phi})\left[\sin\omega t\sin(\omega t + \phi) + \cos\omega t\cos(\omega t + \phi)\right] + (\omega + \dot{\phi})^2\right\} \\ &= \frac{1}{2}mR^2\left[2\omega^2 + 2\omega\dot{\phi} + \dot{\phi}^2 + 2\omega(\omega + \dot{\phi})\cos\phi\right] \end{aligned}$$

The Lagrange equation for ϕ becomes

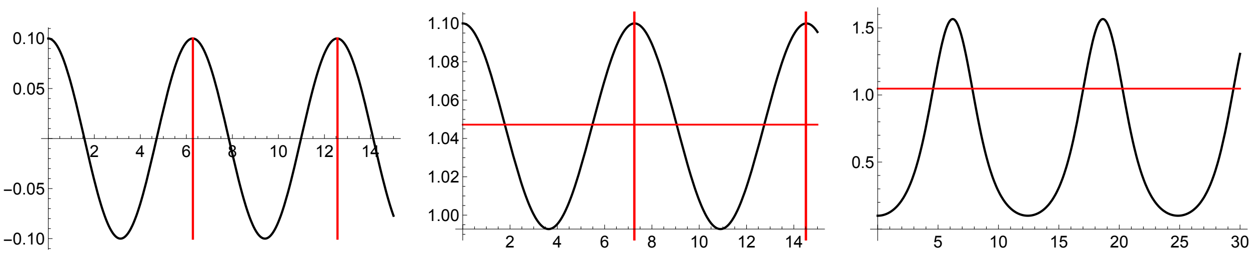
$$\frac{d}{dt}\left[2\omega + 2\dot{\phi} + 2\omega\cos\phi\right] = 2\ddot{\phi} - 2\omega\dot{\phi}\sin\phi = -2\omega(\omega+\dot{\phi})\sin\phi$$

and the differential equation for $\ddot{\phi}$ is

$$\ddot{\phi} = -\omega^2\sin\phi$$

which is the same as the pendulum equation, with small oscillation frequency ω. This makes sense, since in the rotating frame, the centrifugal force at the position of the mass is $m(2R\omega^2)$ which is equivalent to mg in the hanging pendulum, and the length of the pendulum is $\ell = 2R$, so the small oscillation frequency would be $\sqrt{g/\ell} = \sqrt{2R\omega^2/2R} = \omega$.

(5) Equation (7.69) becomes $\theta''(x) = (\cos\theta - \alpha)\sin\theta$. See the accompanying MATHEMATICA notebook for the numerical solution. Following are the plots of the three solutions:



For (a), the differential equation is $\theta''(x) = (\cos\theta - \alpha)\sin\theta = -(\alpha - 1)\theta$ for small θ, so we just have a cosine function with period 2π. For (b), the equilibrium angle is no longer at $\theta = 0$ but instead at $\theta_{\text{eq}} = \cos^{-1}(\alpha) = \pi/3$. However, $\theta_0 - \theta_{\text{eq}}$ is still small, so you expect a cosine function about $\pi/3 = 1.05$. For (c), you are now far away from the equilibrium point, so the oscillations are no longer sinusoidal.

The first two plots show, in vertical red lines, the expected period based on the small displacement approximation. For case (a), the frequency is $\sqrt{\alpha - 1}$. For case (b), we write $\theta = \theta_{\text{eq}} + \epsilon(x)$ where $\epsilon \ll 1$. Then with $\alpha = \cos\theta_{\text{eq}}$ we have

$$\begin{aligned}
[\cos\theta - \cos\theta_{\text{eq}}]\sin\theta &= [\cos(\theta_{\text{eq}}+\epsilon) - \cos\theta_{\text{eq}}]\sin(\theta_{\text{eq}}+\epsilon) \\
&= [\cos\theta_{\text{eq}}\cos\epsilon - \sin\theta_{\text{eq}}\sin\epsilon - \cos\theta_{\text{eq}}][\sin\theta_{\text{eq}}\cos\epsilon + \sin\epsilon\cos\theta_{\text{eq}}] \\
&\approx [\cos\theta_{\text{eq}} - \sin\theta_{\text{eq}}\epsilon - \cos\theta_{\text{eq}}][\sin\theta_{\text{eq}} + \epsilon\cos\theta_{\text{eq}}] \\
\text{so} \qquad \ddot{\epsilon} &= -\sin^2\theta_{\text{eq}}\,\epsilon
\end{aligned}$$

and the frequency is $\sin\theta_{\text{eq}} = \sqrt{1 - \cos^2\theta_{\text{eq}}} = \sqrt{1 - \alpha^2}$.

PHYS3101 Analytical Mechanics   Homework #3   Due 19 Sep 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

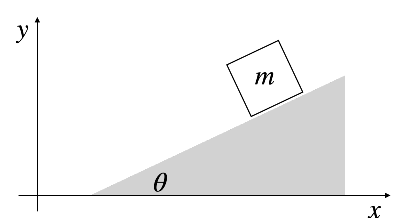
(1) Two equal masses m are connected by a massless string of length L. The string passes through a hole in a frictionless horizontal flat table, so one mass slides freely on the table and the other hangs straight down. Use plane polar coordinates for the mass on the table, and show that the angular coordinate is ignorable. Find a solution to the equation of motion where the radial coordinate can be a constant, and then show that that coordinate is stable under small deviations from that constant.

(2) Consider a system of N different massive particles described by spherical polar coordinate (r, θ, ϕ). Assume that the system is symmetric under rotations about the z-axis. That is, a transformation from (r, θ, ϕ) to $(r, \theta, \phi + \epsilon)$ does not change the Lagrangian. (You can assume that there are no velocity-dependent potential energies.) Determine the associated conserved quantity, and interpret it physically.

(3) Show that the vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ for a uniform, static magnetic field $\mathbf{B}$. Next express $\mathbf{A}$ in cylindrical polar coordinates (ρ, ϕ, z), including its direction with the appropriate unit vector(s). (Use the "smart choice" for the z-direction.) Now write down the Lagrangian and derive the equations of motion for a particle with mass m and charge q in this magnetic field. Describe the solutions of these equations when ρ is a constant. Recall Problem 2 from Homework 1.

(4) Write down the Lagrangian for a simple plane pendulum of length L and bob mass m using Cartesian coordinates (x, y) for the bob. Now write down a suitable constraint equation between x and y. (Many different choices are possible.) Use your constraint equation with a Lagrange multiplier to find modified Lagrange's equations, and show that the result is equivalent to using a single degree of freedom described by the angle θ through which the pendulum swings.

(5) A block of mass m slides down a frictionless inclined plane. Using the x and y coordinates shown, and the method of Lagrange multipliers, find the forces of constraint in the x and y directions. Show that this is just what you expect from your introductory physics course.



(1) Use plane polar coordinates (r,ϕ) of the mass on the table to describe the system, and q be the distance down from the table to the mass that hangs below. Then $q+r=L$, and the Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\dot{q}^2+\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\phi}^2+mgq \\ &= m\dot{r}^2+\frac{1}{2}mr^2\dot{\phi}^2-mgr+\text{constant}\end{aligned}$$

Since $\partial\mathcal{L}/\partial\phi=0$ we know that $\partial\mathcal{L}/\partial\dot{\phi}=mr^2\dot{\phi}\equiv\ell$ is a constant. The r equation is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}}-\frac{\partial\mathcal{L}}{\partial r}=2m\ddot{r}-mr\dot{\phi}^2+mg=2m\ddot{r}-mr\left(\frac{\ell}{mr^2}\right)^2+mg=2m\ddot{r}-\frac{\ell^2}{mr^3}+mg=0$$

We clearly can have $\dot{r}=\ddot{r}=0$ when

$$r^3=\frac{\ell^2}{m^2g}\equiv r_0^3 \qquad \text{that is} \qquad r_0=\left(\frac{\ell^2}{m^2g}\right)^{1/3}$$

Substituting this expression into the equation of motion, we have

$$\ddot{r}=\frac{1}{2}\left[\frac{\ell^2}{m^2r^3}-g\right]=\frac{1}{2}\left[\frac{r_0^3m^2g}{m^2r^3}-g\right]=\frac{g}{2}\left[\frac{r_0^3}{r^3}-1\right]$$

Now put $r(t)=r_0+\epsilon(t)$ where $\epsilon\ll r_0$, and get

$$\ddot{\epsilon}=\frac{g}{2}\left[\frac{r_0^3}{(r_0+\epsilon)^3}-1\right]=\frac{g}{2}\left[\left(1+\frac{\epsilon}{r_0}\right)^{-3}-1\right]\approx\frac{g}{2}\left[1-3\frac{\epsilon}{r_0}-1\right]=-\frac{3}{2}\frac{g}{r_0}\epsilon$$

so we indeed have stable oscillations about the point of equilibrium.

(2) Proceeding exactly as we did for linear translational asymmetry in class, we have

$$\delta\mathcal{L}=\sum_{\alpha=1}^{N}\frac{\partial\mathcal{L}}{\partial\phi_\alpha}\epsilon=\epsilon\frac{d}{dt}\sum_{\alpha=1}^{N}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_\alpha}=0 \qquad \text{so} \qquad \sum_{\alpha=1}^{N}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_\alpha}=\sum_{\alpha=1}^{N}m_\alpha r_\alpha^2\sin^2\theta_\alpha\dot{\phi}_\alpha=\text{constant}$$

(Exercise 4.10 in Concepts shows how to get the differential $d\mathbf{r}$, which gives $\dot{\mathbf{r}}$.) Recognize that $r\sin\theta=\rho$ is just the distance from the z-axis to the particle in question, so the term in the sum is the angular momentum ℓ_α. In other words, rotational symmetry about the z-axis means that the z-component of angular momentum is conserved.

(3) Just take the curl of $\mathbf{A}$ and show that you get $\mathbf{B}$. We have

$$\begin{aligned}\mathbf{A} &= \frac{1}{2}\mathbf{B}\times\mathbf{r}=\frac{1}{2}\hat{\mathbf{x}}(B_yz-B_zy)+\frac{1}{2}\hat{\mathbf{y}}(B_zx-B_xz)+\frac{1}{2}\hat{\mathbf{z}}(B_xy-B_yx) \\ \boldsymbol{\nabla}\times\mathbf{A} &= \hat{\mathbf{x}}\left[\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right]+\hat{\mathbf{y}}\left[\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right]+\hat{\mathbf{z}}\left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right] \\ &= \frac{1}{2}\hat{\mathbf{x}}\left[B_x-(-B_x)\right]+\frac{1}{2}\hat{\mathbf{y}}\left[B_y-(-B_y)\right]+\frac{1}{2}\hat{\mathbf{z}}\left[B_z-(-B_z)\right] \\ &= \hat{\mathbf{x}}B_x+\hat{\mathbf{y}}B_y+\hat{\mathbf{z}}B_z=\mathbf{B}\end{aligned}$$

The only "direction" in the problem is $\mathbf{B}$ so make that the z-direction. Then,

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} = \frac{B}{2}\hat{\mathbf{z}}\times(\hat{\boldsymbol{\rho}}\rho + \hat{\mathbf{z}}z) = \frac{B\rho}{2}\hat{\mathbf{z}}\times(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi) = \frac{B\rho}{2}(\hat{\mathbf{y}}\cos\phi - \hat{\mathbf{x}}\sin\phi) = \frac{B\rho}{2}\hat{\boldsymbol{\phi}}$$

The Lagrangian is therefore

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}}\cdot\mathbf{A} = \frac{1}{2}m(\hat{\boldsymbol{\rho}}\dot{\rho} + \hat{\boldsymbol{\phi}}\rho\dot{\phi} + \hat{\mathbf{z}}\dot{z})^2 + q(\hat{\boldsymbol{\rho}}\dot{\rho} + \hat{\boldsymbol{\phi}}\rho\dot{\phi} + \hat{\mathbf{z}}\dot{z})\cdot\frac{B\rho}{2}\hat{\boldsymbol{\phi}} \\ &= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + \frac{1}{2}qB\rho^2\dot{\phi}\end{aligned}$$

Lagrange's equations become

$$m\ddot{\rho} = m\rho\dot{\phi}^2 + qB\rho\dot{\phi} \qquad \frac{d}{dt}\left(m\rho^2\dot{\phi} + \frac{1}{2}qB\rho^2\right) = 0 \qquad m\ddot{z} = 0$$

The z-equation just says the particle moves with constant velocity in the z-direction, so concentrate on what happens in the plane projection. If $\rho = R$ is a constant, then the middle equation just says that $m\rho^2\dot{\phi}$ is a constant, so $\dot{\phi}$ is constant. The first equation becomes $\dot{\phi}(m\dot{\phi} + qB) = 0$ so either $\dot{\phi} = 0$ and the particle just moves in a straight line in the z-direction, or $\dot{\phi} = -qB/m$ and the particle moves in a circle of radius R at constant angular frequency qB/m. This is the same result we got in Problem 2 of Homework 1.

(4) Take the origin to be the pivot point, with y pointing *downards*. Then, simply,

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy$$

With these coordinates, the constraint equation can be written as

$$f(x,y) = x^2 + y^2 = L^2 \text{ (a constant)}$$

The modified Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial y} + \lambda\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}}$$

become the equations

$$2\lambda x = m\ddot{x} \qquad \text{and} \qquad mg + 2\lambda y = m\ddot{y}$$

If θ measures the angle through which the pendulum swings, with $\theta = 0$ being the pendulum hanging down as usual, then $x = L\sin\theta$ and $y = L\cos\theta$. Using

$$\ddot{x} = \frac{d}{dt}(L\dot{\theta}\cos\theta) = L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta \qquad \text{and} \qquad \ddot{y} = \frac{d}{dt}(-L\dot{\theta}\sin\theta) = -L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta$$

we can equate 2λ from these two equations to find

$$\begin{aligned}m\frac{L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta}{L\sin\theta} &= m\frac{-L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta - g}{L\cos\theta} \\ \ddot{\theta}\cos^2\theta - \dot{\theta}^2\sin\theta\cos\theta &= -\ddot{\theta}\sin^2\theta - \dot{\theta}^2\cos\theta\sin\theta - \omega^2\sin\theta\end{aligned}$$

so $\ddot{\theta} + \omega^2\sin\theta = 0$ where $\omega^2 = g/L$. This is indeed the pendulum equation.

(5) The equation of the plane is something like $y = k(x - x_0)$ where $k = \tan\theta$ and x_0 is some constant. So write the constraint as $f(x, y) = \text{constant}$ where $f(x, y) = y - kx$. The Lagrangian is simple, namely

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

The modified Lagrange equations (see above in Problem (4)) become the equations

$$\lambda(-k) = m\ddot{x} \qquad \text{and} \qquad -mg + \lambda(1) = m\ddot{y}$$

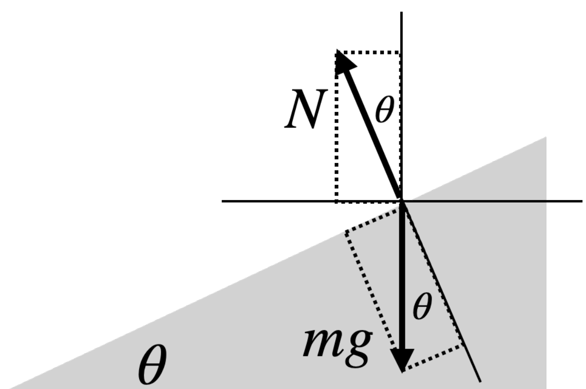
The constraint tells us that $\ddot{y} = k\ddot{x}$, so multiply the first equation by k and subtract to find

$$-k^2\lambda + mg - \lambda = 0 \qquad \text{so} \qquad \lambda = \frac{mg}{1+k^2} = mg\cos^2\theta$$

Therefore, the constraint forces are

$$\begin{aligned} F_x &= \lambda\frac{\partial f}{\partial x} = -k\lambda = -\tan\theta\, mg\cos^2\theta = -mg\cos\theta\sin\theta \\ \text{and} \qquad F_y &= \lambda\frac{\partial f}{\partial y} = \lambda = mg\cos^2\theta \end{aligned}$$

In Introductory Physics, you called the constraints "normal forces" because they were perpendicular to the surface on which the mass sits. The magnitude of the normal force must equal the component of the weight normal to the surface, that is $N = mg\cos\theta$:



Referring to the figure, it is clear that

$$N_x = -N\sin\theta = -mg\cos\theta\sin\theta \qquad \text{and} \qquad N_y = N\cos\theta = mg\cos^2\theta$$

PHYS3101 Analytical Mechanics Homework #4 Due 26 Sep 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) We showed that the Lagrangian for a system of two masses m_1 and m_2, which only interact through a "central" potential $U(|\mathbf{r}_1 - \mathbf{r}_2|)$, decouples into a center-of-mass (CM) coordinate $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$ where $M = m_1 + m_2$, and a relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

(a) Show that the potential energy $U(\mathbf{r}) = -F\mathbf{r}\cdot\hat{\mathbf{n}}$ for a uniform force field $\mathbf{F} = F\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is an arbitrary unit vector.

(b) Show that even in the presence of an external force $\mathbf{F} = ma\hat{\mathbf{n}}$, where a is a constant and m is the mass, the Lagrangian still decouples into CM and relative coordinates.

(c) Explain why the Earth-Moon system can be described (to a very good approximation) as a two-body central force problem, even though both are orbiting about the Sun.

(2) Two bodies with masses m and M orbit each other based on their mutual gravitational attraction. If we don't make the assumption that $m \ll M$ for Keplerian orbits, show that the correct form of Kepler's Third Law is

$$\tau^2 = \frac{4\pi^2}{G(M+m)}a^3$$

for period τ and semimajor axis a. Use an elementary "$F = ma$" approach, where a is the centripetal acceleration, to show that this is correct for two stars of the same mass following circular orbits about their common center of mass.

(3) Calculate the radius of a stable circular orbit by finding the minimum in the effective potential energy for a mass m orbiting a much larger mass M. Use Use an elementary "$F = ma$" approach, where a is the centripetal acceleration, to show that you got the correct answer. Then, calculate the period of small oscillations about this point by expanding the effective potential in a Taylor series, and compare to the period of the circular orbit.

(4) In General Relativity, Newtonian gravity is modified so that the force is

$$F(r) = -G\frac{mM}{r^2}\left(1 - \frac{R_s}{r}\right)$$

where $R_s = 2GM/c^2$ is the "Schwarzschild radius" of the large mass M, and is very much smaller than the orbital distance r for planets in our Solar System. Show that the orbit is very nearly an ellipse (for negative total energy), and calculate by how much the axis of the ellipse precesses (in angle) over one orbit, if the orbit is nearly circular with radius R.

(5) Calculate the amount of time (in years) it would take to launch a spacecraft from Earth to Uranus, using the most direct path possible, using no fuel other than to shoot it out of Earth's orbit. Assume that Earth and Uranus are in circular orbits with radii 1 AU and 19.2 AU. You can also assume that the launch happens at the right time so that Uranus is in the right place when the spacecraft arrives at its orbit.

(1) For (a), just take the gradient of the given function and find

$$-\boldsymbol{\nabla}U = F\boldsymbol{\nabla}(xn_x + yn_y + zn_z) = F(\hat{\mathbf{x}}n_x + \hat{\mathbf{y}}n_y + \hat{\mathbf{z}}n_z) = F\hat{\mathbf{n}}$$

The kinetic energy decouples the same as before, namely as given in Taylor (8.10) through (8.12), and the internal potential energy is unchanged. The new term in the Lagrangian is

$$m_1 a\mathbf{r}_1 \cdot \hat{\mathbf{n}} + m_2 a\mathbf{r}_2 \cdot \hat{\mathbf{n}} = Ma\mathbf{R} \cdot \hat{\mathbf{n}}$$

which depends only on $\mathbf{R}$ and not $\mathbf{r}$. That is, there is a force Ma acting on the center of mass, but the motion in the relative coordinate is unchanged. This applies to the Earth-Moon system because their separation is very much smaller than the distance to the Sun, so the gravitational attraction of the Sun is effectively in the same direction for both, and proportional to the mass of each. In other words, the CM revolved around the Sun, while the Earth and Moon revolve about their mutual center of mass.

(2) From Taylor (8.54) and the text that follows, Kepler's Third Law is

$$\tau^2 = 4\pi^2 \frac{a^3\mu}{\gamma} = 4\pi^2 a^3 \frac{mM}{m+M}\frac{1}{GmM} = \frac{4\pi^2}{G(M+m)}a^3 = \frac{2\pi^2}{GM}a^3$$

if $m = M$. For a circular orbit, "a" is the radius in the relative coordinate frame, namely the separation between the two masses. That is, $a = 2R$ where R is the radius of the circular orbit of each of the two masses about their common center of mass, which is exactly halfway between them. That is

$$\tau^2 = \frac{16\pi^2}{GM}R^3$$

For circular orbits of radius R, the force between the two bodies is $GM^2/(2R)^2$. The orbital velocity $v = 2\pi R/\tau$ and the centripetal acceleration of each body is v^2/R. Newton's Second Law on either body gives

$$\frac{GM^2}{4R^2} = M\frac{v^2}{R} = M\frac{1}{R}\frac{4\pi^2 R^2}{\tau^2} \qquad \text{so} \qquad \tau^2 = \frac{16\pi^2}{GM}R^3$$

which is the same answer we arrived at above.

(3) The effective potential is given by Taylor (8.32) as

$$U_{\text{eff}}(r) = -G\frac{mM}{r} + \frac{\ell^2}{2\mu r^2} \qquad \text{where} \qquad \ell = \mu r^2\dot{\phi} \qquad \text{and} \qquad \mu = \frac{mM}{m+M} \approx m$$

We differentiate with respect to r to find the minimum:

$$\frac{d}{dr}\left[-G\frac{mM}{r} + \frac{\ell^2}{2mr^2}\right] = G\frac{mM}{r^2} - \frac{\ell^2}{mr^3} = 0 \qquad \text{so} \qquad r = \frac{\ell^2}{Gm^2M} \equiv R$$

is the radius of the circular orbit. Writing $\omega = \dot{\phi}$, a constant for this orbit,

$$R = \frac{m^2R^4\omega^2}{Gm^2M} \qquad \text{so} \qquad R = \left(\frac{GM}{\omega^2}\right)^{1/3}$$

In a Physics I course, you would have written that the centripetal acceleration is $\omega^2 R$ and the attractive force to be GmM/R^2 so

$$G\frac{mM}{R^2} = m\omega^2 R \qquad \text{and} \qquad R^3 = \frac{GM}{\omega^2}$$

which is the same as the answer we got the fancy way.

To find the period of small oscillations about this minimum, and to confirm that the orbit is stable, we expand the effective potential in Taylor series about $r = R$ to get

$$U_{\text{eff}}(r) = U_{\text{eff}}(R) + \frac{d}{dr}U_{\text{eff}}(r)\bigg|_{r=R}(r-R) + \frac{1}{2}\frac{d^2}{dr^2}U_{\text{eff}}(r)\bigg|_{r=R}(r-R)^2 + \cdots$$

The first term is just a constant, and the second term is zero because that's how we determined R. The third term is a simple harmonic oscillator potential with effective spring constant

$$k = \frac{d^2}{dr^2}U_{\text{eff}}(r)\bigg|_{r=R} = -2G\frac{mM}{R^3} + 3\frac{\ell^2}{mR^4} = -2GmM\frac{\omega^2}{GM} + 3\frac{m^2R^4\omega^2}{mR^4} = m\omega^2$$

Therefore, the frequency of these small oscillations is just $\omega = v/R$ where $v = \dot{\phi}$ is the (constant) velocity of the mass m in a circular orbit. That is, the period of small oscillations is in fact just "one year", as expected.

(4) Following the problem statement in Taylor 8.23, write

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3} = -G\frac{mM}{r^2} + G\frac{mM}{r^2}\frac{R_s}{r} \qquad \text{so} \qquad k = GmM \qquad \text{and} \qquad \lambda = GmMR_s$$

The differential equation for the orbit $u(r) = 1/r$ in (8.41) becomes

$$u''(\phi) = -u(\phi) + \frac{\mu}{\ell^2}k - \frac{\mu}{\ell^2}\lambda u(\phi) = -\left(1 + \frac{\mu\lambda}{\ell^2}\right)u(\phi) + \frac{\mu k}{\ell^2} = -\beta^2 u(\phi) + \frac{\mu k}{\ell^2}$$

where we have made the definition

$$\beta \equiv \left(1 + \frac{\mu\lambda}{\ell^2}\right)^{1/2} \approx 1 + \frac{\mu\lambda}{2\ell^2}$$

where we anticipate that $\mu\lambda/\ell^2 \ll 1$. Following Taylor, we write the solution as

$$u(\phi) = A\cos(\beta\phi - \delta) + \frac{\mu k}{\beta^2\ell^2} = \frac{1}{c}(1 + \epsilon\cos\beta\phi) \qquad c = \frac{\beta^2\ell^2}{\mu k} \qquad \epsilon = \frac{A\beta^2\ell^2}{\mu k}$$

where we choose the arbitrary phase $\delta = 0$. That is, the orbit is described in plane polar coordinates as

$$r = \frac{c}{1 + \epsilon\cos(\beta\phi)}$$

which is very close to an ellipse since $\beta \approx 1$. If α is the precession over one orbit, then the argument of the cosine reaches 2π when ϕ reaches $2\pi + \alpha$, so $\beta(2\pi + \alpha) = 2\pi$. For a nearly circular orbit with radius R, $\ell = \mu\omega R^2$ and taking $m \ll M$ so that $\mu \approx m$,

$$\alpha = 2\pi\left(1 - \frac{1}{\beta}\right) = \frac{\pi\mu\lambda}{\ell^2} = \frac{\pi\mu GmMR_s}{\mu^2\omega^2R^4} = \frac{\pi GMR_s}{\omega^2R^4} = \frac{\pi c^2}{2\omega^2R^2}\left(\frac{R_s}{R}\right)^2$$

which is clearly dimensionless, and a very small number for the Sun-Mercury system.

I'd like to have asked them to put in numbers and show that it is in fact the 43 seconds of arc, but not enough time for me to work it out for myself.

(5) We launch the spacecraft out of Earth's orbit at a speed that gives it an elliptical trajectory whose perihelion is at the orbit of Uranus. The radius of Earth's orbit is R_1 and the radius of Uranus' orbit is R_2. Therefore, the semi major axis of the elliptical transfer orbit is

$$a_T = \frac{1}{2}(R_1 + R_2) = 10.1 \text{ AU}$$

We can scale from the length of time that it takes for an Earth orbit (that is, one year) to that for the transfer orbit, just by scaling using Kepler's Third Law. That is

$$\left(\frac{\tau_T}{\tau_E}\right)^2 = \left(\frac{a_T}{a_E}\right)^3$$

Measuring time in years and distances in AU, we get

$$\tau_T = a_T^{3/2} = 32 \text{ years}$$

The spacecraft only uses half an orbit to get to Uranus, so the travel time is 16 years.

PHYS3101 Analytical Mechanics Homework #5 Due 3 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A mass m follows an elliptical orbit for a potential energy $U(r) = -\gamma/r$. Show that

$$\boldsymbol{\epsilon} \equiv \frac{1}{\gamma}\dot{\mathbf{r}} \times \boldsymbol{\ell} - \hat{\mathbf{r}}$$

where $\boldsymbol{\ell} = \mathbf{r} \times m\dot{\mathbf{r}}$, is a (vector) constant of the motion, and that its magnitude is the eccentricity of the ellipse. What physical characteristic of the ellipse is described by the *direction* of $\boldsymbol{\epsilon}$? *Hints and comments:* Make use of vector identities with cross products, including $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$, and $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. To find $\epsilon = |\boldsymbol{\epsilon}|$ do $\epsilon^2 = \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}$, assemble terms, and recognize the total energy. For the direction, use a simple dot product to show that $\boldsymbol{\epsilon}$ lies in the plane of the orbit. Then, evaluate $\boldsymbol{\epsilon} \cdot \mathbf{r}$ assuming that the angle between $\boldsymbol{\epsilon}$ and $\mathbf{r}$ is α. Solve for r, and compare to the formula for an ellipse to discover the relationship between α and ϕ. (Your instincts could tell you the direction of $\boldsymbol{\epsilon}$; the "obvious" choice turns out to be correct.)

(2) A 120 lb person sits in a Chevy Corvette, which accelerates from zero to 60 miles per hour in six seconds. (That's faster than most cars.) Assuming the acceleration is constant, how much force (in pounds) does she feel pushing her against the back of the seat? A weight hangs from a string suspended from the ceiling. What angle does it make with the vertical?

(3) An observer sits on a turntable which rotates counter clockwise at a constant angular speed Ω. A mass m rides on the frictionless surface of the turntable. The observer sees the mass move in a circle of radius R at fixed angular velocity ω about the same axis as the turntable. Find the value of ω such that the combined centrifugal and Coriolis forces give just the right centripetal force $m\omega^2 R$, directed towards the center of rotation, to maintain the circular motion. What's going on? (The answer should be obvious.)

(4) On a certain planet, which is perfectly spherically symmetric, the free fall acceleration has magnitude $g = g_0$ at the North Pole and $g = \lambda g_0$ at the equator (with $0 \leq \lambda \leq 1$). Find $g(\theta)$, the free fall acceleration at a colatitude θ as a function of θ.

(5) Use the method of successive approximations to find the path $\mathbf{r}(t)$, to first order in the Earth's rotation speed Ω, of an object thrown from an origin located at colatitude θ with initial velocity $\mathbf{v}_0 = v_{x_0}\hat{\mathbf{x}} + v_{y_0}\hat{\mathbf{y}} + v_{z_0}\hat{\mathbf{z}}$. Assume that the acceleration vector $\mathbf{g}$ due to gravity is constant throughout the flight, and ignore air resistance.

PHYS3101 Analytical Mechanics Homework #5 Solutions

(1) With $\boldsymbol{\epsilon} \equiv \dot{\mathbf{r}} \times \boldsymbol{\ell}/\gamma - \hat{\mathbf{r}}$ and $\boldsymbol{\ell} = \mathbf{r} \times m\dot{\mathbf{r}}$, use $\hat{\mathbf{r}} = \mathbf{r}/r$ and $m\ddot{\mathbf{r}} = -(\gamma/r^3)\mathbf{r}$ to write

$$\frac{d\boldsymbol{\epsilon}}{dt} = \frac{1}{\gamma}\ddot{\mathbf{r}} \times \boldsymbol{\ell} - \frac{\dot{\mathbf{r}}}{r} + \frac{\mathbf{r}}{r^2}\dot{r} = -\frac{1}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) - \frac{\dot{\mathbf{r}}}{r} + \frac{\mathbf{r}}{r^2}\dot{r} = -\frac{\mathbf{r}}{r^3}\mathbf{r} \cdot \dot{\mathbf{r}} + \frac{\dot{\mathbf{r}}}{r^3}r^2 - \frac{\dot{\mathbf{r}}}{r} + \frac{\mathbf{r}}{r^2}\dot{r} = 0$$

For magnitude, consider $\epsilon^2 = (\dot{\mathbf{r}} \times \boldsymbol{\ell})^2/\gamma^2 - 2\hat{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \boldsymbol{\ell})/\gamma + 1$. To evaluate the first term, use $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$. So, $(\dot{\mathbf{r}}\times\boldsymbol{\ell})^2 = \dot{\mathbf{r}}^2\ell^2 - (\dot{\mathbf{r}}\cdot\boldsymbol{\ell})^2 = \dot{\mathbf{r}}^2\ell^2$. For the second term, use "BAC-CAB" to write $\dot{\mathbf{r}} \times \boldsymbol{\ell} = m\dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}}) = m\mathbf{r}\dot{\mathbf{r}}^2 - m\dot{\mathbf{r}}(\dot{\mathbf{r}} \cdot \mathbf{r}) = m\mathbf{r}\dot{\mathbf{r}}^2 - mr\dot{r}\dot{\mathbf{r}}$. So, $\hat{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \boldsymbol{\ell}) = mr\dot{\mathbf{r}}^2 - mr\dot{r}^2 = mr^3\dot{\phi}^2 = \ell^2/mr$, and

$$\epsilon^2 = \frac{1}{\gamma^2}\dot{\mathbf{r}}^2\ell^2 - \frac{2}{\gamma}\frac{\ell^2}{mr} + 1 = \frac{2\ell^2}{m\gamma^2}\left[\frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{\gamma}{r}\right] + 1 = \frac{2\ell^2 E}{m\gamma^2} + 1$$

This agrees with Taylor (8.58). For the direction, use a little trickery. Since $\boldsymbol{\ell} \propto \hat{\mathbf{z}}$, $\boldsymbol{\epsilon} \cdot \boldsymbol{\ell} = 0$. Now, let α be the angle between $\boldsymbol{\epsilon}$ and $\mathbf{r}$, so $\boldsymbol{\epsilon} \cdot \mathbf{r} = \epsilon r \cos\alpha = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \boldsymbol{\ell})/\gamma - r$. Then use the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ to write $\mathbf{r} \cdot (\dot{\mathbf{r}} \times \boldsymbol{\ell}) = \boldsymbol{\ell} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \boldsymbol{\ell} \cdot \boldsymbol{\ell}/m = \ell^2/m$. Therefore

$$\epsilon r \cos\alpha = \frac{\ell^2}{m\gamma} - r \qquad \text{and, so,} \qquad r = \frac{\ell^2/m\gamma}{1 + \epsilon\cos\alpha}$$

which is $r = r(\phi)$ for an ellipse, with $\phi = \alpha$, i.e., $\boldsymbol{\epsilon}$ lies along the major axis of the ellipse.

(2) Convert to SI and back, using "pounds" as force or mass depending on the context. Then $m = 120/2.205 = 54.4$ kG. Have 60 mph$= 60 \times 0.447 = 26.8$ m/s, so the horizontal acceleration $A = 26.8/5 = 5.4$ m/s^2 $= 0.55g$. The "force" is $mA = 5.4 \times 54.4 = 294$ N which is equivalent to $294 \times 0.2248 = 66 (= 0.55 \cdot 120)$ pounds. Gravity is down and the car accelerates horizontally, so $\theta = \tan^{-1}(A/g)\tan^{-1}(0.55) = 0.50$ rad$= 29°$.

(3) Have $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ for counterclockwise rotation. With cylindrical polar coordinates (r, ϕ, z), $\mathbf{r} = R\hat{\mathbf{r}}$ and $\dot{\mathbf{r}} = R\omega\hat{\boldsymbol{\phi}}$ for the mass, so $\mathbf{F}_{\text{cf}} = m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} = m\Omega R\hat{\boldsymbol{\phi}} \times \boldsymbol{\Omega} = m\Omega^2\mathbf{r}$ and then $\mathbf{F}_{\text{Cor}} = 2m\dot{\mathbf{r}}\times\boldsymbol{\Omega} = 2m\omega\Omega\mathbf{r}$. With $\mathbf{F}_{\text{cf}} + \mathbf{F}_{\text{Cor}} = -m\omega^2 R\hat{\mathbf{r}}$ have $\omega^2 + 2m\omega\Omega + \Omega^2 = (\omega+\Omega)^2 = 0$. Therefore $\omega = -\Omega$. The mass rotates in the opposite direction to the turntable but with the same angular speed. To an observer off the turntable, the mass is simply standing still.

(4) From Taylor (9.44), and referring to Figure 9.10, $\mathbf{g} = \mathbf{g}_0 + \Omega^2 R\sin\theta\hat{\boldsymbol{\rho}}$. At the pole, $\theta = 0$ and $g = g_0$. At the equator, $\theta = 90°$ and $g(90°) = g_0 - \Omega^2 R = \lambda g_0$, giving $\Omega^2 R = g_0(1 - \lambda)$. Now $g^2 = g_0^2 + 2\Omega^2 R\sin\theta\mathbf{g}_0\cdot\hat{\boldsymbol{\rho}} + \Omega^4 R^2\sin^2\theta = g_0^2[1 + 2(1-\lambda)\sin\theta\cos(\pi/2+\theta) + (1-\lambda)^2\sin^2\theta] = g_0^2[1 - \sin^2\theta + \lambda^2\sin^2\theta] = g_0^2[\cos^2\theta + \lambda^2\sin^2\theta]$ and so $g = g_0[\cos^2\theta + \lambda^2\sin^2\theta]^{1/2}$.

(5) The zeroth order solutions are $\dot{x} = v_{0x}$, $\dot{y} = v_{0y}$, and $\dot{z} = v_{0z} - gt$. Inserting this back into (9.53) gives us the first order equations we need to solve, namely

$$\begin{aligned}
\ddot{x} &= 2\Omega(v_{0y}\cos\theta - v_{0z}\sin\theta) + 2\Omega gt\sin\theta \\
\ddot{y} &= -2\Omega v_{0x}\cos\theta \\
\ddot{z} &= -g + 2\Omega v_{0x}\sin\theta
\end{aligned}$$

These are now simple uncoupled second order equations with known initial conditions. Integrating them all the way is simple, and gives

$$\begin{aligned}
x(t) &= v_{x_0}t + \Omega(v_{y_0}\cos\theta - v_{z_0}\sin\theta)t^2 + \frac{1}{3}\Omega gt^3\sin\theta \\
y(t) &= v_{y_0}t - \Omega(v_{x_0}\cos\theta)t^2 \\
z(t) &= v_{z_0}t - \frac{1}{2}gt^2 + \Omega(v_{x_0}\sin\theta)t^2
\end{aligned}$$

The answer is in fact given in the problem statement of Taylor 9.26.

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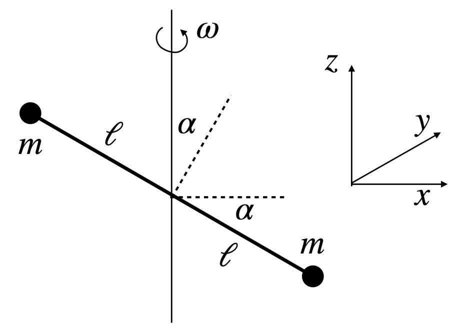
PHYS3101 Analytical Mechanics   Homework #6   Due 10 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) The figure shows two equal masses m at the ends of a massless rod of length 2ℓ, rotating with angular velocity ω about an axis which passes through the center of mass. The normal vector to the rod makes an angle α with respect to the axis of rotation. At the instant shown, the rod lies in the xz plane. Use the coordinate system shown for the following calculations:



(a) Find all nine components of the inertia tensor for this coordinate system. **(b)** Find the (vector) angular momentum for the configuration as shown. **(c)** Find the kinetic energy for the configuration as shown. **(d)** Calculate the principal moments of inertia, and **(e)** find the principal axes for this configuration.

(2) Find the moment of inertia about the z-axis for a uniform ellipsoid whose surface is given by $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.(You can use MATHEMATICA if you want, but it's probably easier to just do the necessary integrals with a simple change of variables and exploiting symmetry.) Check your answer against the result for a sphere with $a = b = c = R$.

(3) Consider a top consisting of a uniform cone spinning freely about its tip at 1800 rpm. If its height is 10 cm and its base radius 2.5 cm, at what angular velocity will it precess?

(4) A rigid body is rotating freely, subject to zero torque. Use Euler's equations to prove that the magnitude of the angular momentum $\mathbf{L}$ is constant. Similarly, show that the kinetic energy of rotation

$$T_{\text{rot}} = \frac{1}{2}\lambda_1\omega_1^2 + \frac{1}{2}\lambda_2\omega_2^2 + \frac{1}{2}\lambda_3\omega_3^2$$

where the λ_i are the principle moments of inertia, is a constant of the motion.

(5) You are probably aware that the Earth's axis of rotation precesses slowly, so that, far in the future, the North pole will no longer be pointing at Polaris. To gain an understanding of this phenomenon, imagine that the Earth is perfectly rigid, uniform, and spherical and is spinning about its usual axis at its usual rate. A huge mountain of mass 10^{-8} Earth masses is now added at colatitude 60°, causing the earth to begin free precession as discussed in class. How long will it take the North Pole (defined as the northern end of the diameter along $\boldsymbol{\omega}$) to move 100 miles from its current position?

(1) The inertia tensor components are $I_{ij} = \sum m_\alpha(r_\alpha^2\delta_{ij} - r_{i_\alpha}r_{j_\alpha})$ where, for the two particles with mass $m_\alpha = m$, $r_\alpha = \ell$, $x_\alpha = \mp\ell\cos\alpha$, $y_\alpha = 0$, and $z_\alpha = \pm\ell\sin\alpha$. Therefore, the diagonal elements are $I_{xx} = 2m\ell^2(1-\cos^2\alpha) = 2m\ell^2\sin^2\alpha$, $I_{yy} = 2m\ell^2$, and $I_{zz} = 2m\ell^2\cos^2\alpha$. (The value of I_{yy} is obvious, and I_{xx} and I_{zz} are correct in the limits $\alpha = 0$ and $\alpha = \pi/2$.) Also $I_{xy} = I_{yx} = 0 = I_{yz} = I_{zy}$ and $I_{xz} = 2m\ell^2\sin\alpha\cos\alpha = m\ell^2\sin(2\alpha) = I_{zx}$. In matrix form,

$$\mathsf{I} = \begin{bmatrix} 2m\ell^2\sin^2\alpha & 0 & m\ell^2\sin(2\alpha) \\ 0 & 2m\ell^2 & 0 \\ m\ell^2\sin(2\alpha) & 0 & 2m\ell^2\cos^2\alpha \end{bmatrix} = 2m\ell^2 \begin{bmatrix} \sin^2\alpha & 0 & \sin\alpha\cos\alpha \\ 0 & 1 & 0 \\ \sin\alpha\cos\alpha & 0 & \cos^2\alpha \end{bmatrix}$$

The principal moments of inertia are given by the eigenvalues, and the principle axes are given by the eigenvectors. The characteristic equation for the matrix above is

$$(\sin^2\alpha - \lambda)(1-\lambda)(\cos^2\alpha - \lambda) - \sin^2\alpha\cos^2\alpha = -\lambda(\lambda^2 - 2\lambda + 1) = 0$$

so the principle moments of inertia are 0, and (twice) $2m\ell^2$, which we could have guessed. For $\lambda = 0$ we have

$$\begin{bmatrix} \sin^2\alpha & 0 & \sin\alpha\cos\alpha \\ 0 & 1 & 0 \\ \sin\alpha\cos\alpha & 0 & \cos^2\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

so that $a\sin\alpha + c\cos\alpha = 0$ or $\hat{\mathbf{n}} = \hat{\mathbf{x}}\cos\alpha - \hat{\mathbf{z}}\sin\alpha$ is the direction of the principle axis. This points along the axis of the rod, as we expect. For $\lambda = 1$, we have

$$\begin{bmatrix} \sin^2\alpha - 1 & 0 & \sin\alpha\cos\alpha \\ 0 & 0 & 0 \\ \sin\alpha\cos\alpha & 0 & \cos^2\alpha - 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\cos^2\alpha & 0 & \sin\alpha\cos\alpha \\ 0 & 0 & 0 \\ \sin\alpha\cos\alpha & 0 & -\sin^2\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

which is satisfied by $b = 1$ or $a\cos\alpha - c\sin\alpha = 0$. In the first case, the principle axis is $\hat{\mathbf{n}} = \hat{\mathbf{y}}$, and in the second case $\hat{\mathbf{n}} = \hat{\mathbf{x}}\sin\alpha - \hat{\mathbf{z}}\cos\alpha$ which is perpendicular to the rod. Both are obviously correct. The angular momentum $\mathbf{L} = \mathsf{I}\cdot\boldsymbol{\omega}$, again using matrices, is

$$2m\ell^2 \begin{bmatrix} \sin^2\alpha & 0 & \sin\alpha\cos\alpha \\ 0 & 1 & 0 \\ \sin\alpha\cos\alpha & 0 & \cos^2\alpha \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = 2m\ell^2 \begin{bmatrix} \omega\sin\alpha\cos\alpha \\ 0 \\ \omega\cos^2\alpha \end{bmatrix}$$

so that $\mathbf{L} = 2m\ell^2\omega\cos\alpha(\hat{\mathbf{x}}\sin\alpha + \hat{\mathbf{z}}\cos\alpha)$, notably not parallel to $\boldsymbol{\omega}$ unless $\alpha = 0$. The kinetic energy $T = \boldsymbol{\omega}\cdot\mathbf{L} = m\ell^2\omega^2\cos^2\alpha = \frac{1}{2}(2m\ell^2)\omega^2 \times \cos^2\alpha$ which, again and finally, is obviously correct for $\alpha = 0$ and $\alpha = \pi/2$.

(2) Use coordinates $\xi \equiv x/a$, $\eta \equiv y/b$, and $\zeta \equiv z/c$, so the limits of integration are all -1 to 1, and the equation of the ellipsoid is $\xi^2 + \eta^2 + \zeta^2 = 1$, that is, a unit sphere. Therefore

$$I = \int (x^2+y^2)dm = \frac{M}{(4/3)\pi abc} abc \int_{\text{sphere}} (a^2\xi^2 + b^2\eta^2)dV = \frac{3M}{4\pi}(a^2+b^2)\frac{4\pi}{15} = \frac{M}{5}(a^2+b^2)$$

where $\int_{\text{sphere}} \xi^2 dV = \int_{\text{sphere}} \eta^2 dV = \int_{\text{sphere}} z^2 dV = \int r^2\cos^2\theta dV = 2\pi(1/5)(2/3) = 4\pi/15$. This obviously gives the correct answer when $a = b (= c) = R$.

(3) This problem is a simple application of (10.83) in Taylor, along with the moment of inertia calculated in Example 10.3. However, we also need to calculate the position of the center of mass of the cone. Referring to Figure 10.6, we have

$$\begin{aligned} z_{CM} &= \frac{1}{M}\int_0^h z\,dm = \frac{1}{M}\int_0^h z\frac{3M}{\pi R^2 h}\,\pi r^2\,dz = \frac{1}{M}\int_0^h z\frac{3M}{\pi R^2 h}\,\pi\left(\frac{Rz}{h}\right)^2\,dz \\ &= \frac{3}{h^3}\int_0^h z^3\,dz = \frac{3}{4}h \end{aligned}$$

Therefore, the precession frequency becomes

$$\Omega = \frac{Mgz_{CM}}{I_{zz}\omega} = \frac{Mg}{\omega}\frac{3}{4}h\frac{10}{3MR^2} = \frac{5}{2}\frac{gh}{\omega R^2}$$

Using $g = 9.8$ m/s^2, $h = 0.1$ m, $R = 0.025$ m, and $\omega = 2\pi \times (1800/60)$/s, find

$$\Omega = 20.8/\text{s} = 199 \text{ rpm}$$

(4) Euler's equations, for zero torque, are

$$\begin{aligned} \Gamma_1 &= 0 = \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 \\ \Gamma_2 &= 0 = \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 \\ \Gamma_3 &= 0 = \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 \end{aligned}$$

We want to show that $|\mathbf{L}| = [\mathbf{L}^2]^{1/2} = [L_1^2 + L_2^2 + L_3^2]^{1/2}$ is constant. Its time derivative is

$$\frac{d}{dt}|\mathbf{L}| = \frac{1}{|\mathbf{L}|}\left(L_1\dot{L}_1 + L_2\dot{L}_2 + L_3\dot{L}_3\right)$$

However $L_i = \lambda_i\omega_i$ $(i = 1, 2, 3)$ so $\dot{L}_i = \lambda_i\dot{\omega}_i$ and, making use of Euler's equations, we have

$$\begin{aligned} L_1\dot{L}_1 + L_2\dot{L}_2 + L_3\dot{L}_3 &= \lambda_1^2\omega_1\dot{\omega}_1 + \lambda_2^2\omega_2\dot{\omega}_2 + \lambda_3^2\omega_3\dot{\omega}_3 \\ &= \lambda_1(\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3 + \lambda_2(\lambda_3 - \lambda_1)\omega_1\omega_2\omega_3 + \lambda_3(\lambda_1 - \lambda_2)\omega_1\omega_2\omega_3 \\ &= (\lambda_1\lambda_2 - \lambda_1\lambda_3 + \lambda_2\lambda_3 - \lambda_2\lambda_1 + \lambda_3\lambda_1 - \lambda_3\lambda_2)\omega_1\omega_2\omega_3 \\ &= 0 \end{aligned}$$

and so $d|\mathbf{L}|/dt = 0$ and the magnitude of $\mathbf{L}$ does not change. Similarly

$$\begin{aligned} \frac{dT}{dt} &= \lambda_1\omega_1\dot{\omega}_1 + \lambda_2\omega_2\dot{\omega}_2 + \lambda_3\omega_3\dot{\omega}_3 \\ &= (\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3 + (\lambda_3 - \lambda_1)\omega_1\omega_2\omega_3 + (\lambda_1 - \lambda_2)\omega_1\omega_2\omega_3 \\ &= (\lambda_2 - \lambda_3 + \lambda_3 - \lambda_1 + \lambda_1 - \lambda_2)\omega_1\omega_2\omega_3 \\ &= 0 \end{aligned}$$

(5) This is a problem concerning "free precession" for a body with two equal principal moments. For a sphere of radius R and mass M, these two equal moments are $\lambda_0 = 2MR^2/5 + mR^2$, where the mass of the mountain is m. The moment of inertia for the principle axis through the mountain is just $\lambda = 2MR^2/5$, ignoring the displacement of the center of mass from the center of the sphere. The angle between the Earth's rotation axis (along $\boldsymbol{\omega}$) and the $\hat{\mathbf{e}}_3$ axis is the colatitude θ of the mountain. Therefore, from (10.93), the precession frequency is

$$\Omega_b = \frac{\lambda_0 - \lambda}{\lambda_0}\omega_3 \approx \frac{5}{2}\frac{m}{M}\omega\cos\theta = \frac{5}{2}10^{-8}\frac{2\pi}{1\text{ day}}\frac{1}{2} = \frac{2\pi}{(4/5)\times 10^8\text{ days}}$$

where we ignore the difference between λ_0 and λ in the denominator. Thus, there is one full precession of $\boldsymbol{\omega}$ about $\hat{\mathbf{e}}_3$ in $2\pi/\Omega_b = 8\times 10^7$ days. The precession follows a circle with circumference $2\pi R\sin\theta = 2.1766\times 10^4$ miles. So, to move 100 miles takes $(100/2.1766\times 10^4)\times 8\times 10^7 = 367,546$ days=1006 years.

PHYS3101 Analytical Mechanics Homework #7 Due 17 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Prove that the principle moments of inertial λ_1, λ_2, and λ_3 of any rigid body must satisfy $\lambda_3 \leq \lambda_1 + \lambda_2$. If $\lambda_3 = \lambda_1 + \lambda_2$, what does that imply about the shape of the body? Pick a specific example to check your answer.

(2) A thin, flat, uniform rectangular plate with mass M lies in the xy plane with two of its corners at $(a, b, 0)$ and the origin. Find the plate's inertia tensor and then diagonalize it to find the principle moments of inertia and the principle axes. Comment on the comparison between $\lambda_1 + \lambda_2$ and λ_3, based on Problem **(1)** above. Choose values for a and b and draw diagrams that show the principle axes for the plate if $a = b$ (a square) and $a \neq b$.

(3) For a symmetric rigid body rotating in free space with no external torques, we showed in class that Euler's Equations implied that, in the body frame (for example, the Earth in Problem 5 of Homework #6) both the angular velocity vector $\boldsymbol{\omega}$ and the angular momentum vector $\mathbf{L}$ precessed around the symmetry axis $\hat{\mathbf{e}}_3$ with a frequency $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1$. Now find the the space frame frequency Ω_s at which $\boldsymbol{\omega}$ and $\hat{\mathbf{e}}_3$ precess about $\mathbf{L}$ is $\Omega_s = L/\lambda_1$. See Figure 10.9 in Taylor. You can do this by first explaining why $\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_b + \boldsymbol{\omega}$. Then consider the angles between $\hat{\mathbf{e}}_3$ and $\boldsymbol{\omega}$, and between $\hat{\mathbf{e}}_3$ and $\mathbf{L}$.

(4)A symmetric top of mass M spins about its symmetry axis at an angular speed ω_3, with a fixed point at the origin. The distance from the origin to the center of mass is R. If the top precesses at a fixed angle θ, show that

$$\lambda_1 \Omega^2 \cos\theta - \lambda_3 \omega_3 \Omega + MgR = 0$$

where λ_1 and λ_3 are principle moments of inertia, and Ω is the rate of precession. Assuming that ω_3 is "very large", solve this quadratic equation for the two possible values of Ω. What kind of motions do these two solutions represent? What does "very large" mean for ω_3? That is, very large compared to what?

(5) The effective potential energy for a spinning symmetric top is

$$U_{\text{eff}}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$

where θ is the polar angle from the vertical, L_z and L_3 are the angular momenta vertical and symmetry body axis, respectively, and λ_1 and λ_3 are principle moments of inertia, and M is the mass. The fixed point is at the origin, and R is the distance from the origin to the center of mass. Why is the second term unimportant for understanding the motion of the top? Plot $U_{\text{eff}}(\theta)$ for $\lambda_1 = 1 = MgR$, $L_z = 8$, and $L_3 = 10$, and find to three significant figures the value θ_0 at which the top precesses with constant θ. Find the rate $\Omega \equiv \dot{\phi}$ of steady precession from the equation for L_z, and compare to the approximate result you obtain in the case where the top is spinning "very rapidly"; see Problem **(4)** above.

(1) In the body frame, the principle moments of inertia are

$$\lambda_1 = \int (r_2^2 + r_3^2)\, dm \qquad \lambda_2 = \int (r_1^2 + r_3^2)\, dm \qquad \lambda_3 = \int (r_1^2 + r_2^2)\, dm$$

Therefore

$$\lambda_1 + \lambda_2 = \int (r_2^2 + r_3^2 + r_1^2 + r_3^2)\, dm = \int (r_1^2 + r_2^2)\, dm + \int 2r_3^2\, dm \geq \int (r_1^2 + r_2^2)\, dm = \lambda_3$$

since the integral over a body of a positive definite quantity most be positive. To satisfy the equality, we obviously need

$$\int r_3^2\, dm = 0$$

This can only happen if the body is flat, and lies in the $(1, 2)$ plane, in which case $r_3 = 0$ for all points in the body. We can check this for a flat rectangular plate with mass M and dimensions $a \times b$ and center at the origin. It's easy enough to look up that, in this case,

$$\lambda_1 = \frac{1}{12}Ma^2 \qquad \lambda_2 = \frac{1}{12}Mb^2 \qquad \text{and} \qquad \lambda_3 = \frac{1}{12}M(a^2 + b^2) = \lambda_1 + \lambda_2$$

(2) The calculation of the inertia tensor is straightforward. Just follow the formula, with $z = 0$ everywhere inside the body:

$$\begin{aligned}
I_{xx} &= \int y^2\, dm = \frac{M}{ab}\int_0^a dx \int_0^b dy\, y^2 = \frac{M}{ab}a\frac{b^3}{3} = \frac{1}{3}Mb^2 \\
I_{yy} &= \int x^2\, dm = \frac{M}{ab}\int_0^a x^2\, dx \int_0^b dy = \frac{M}{ab}\frac{a^3}{3}b = \frac{1}{3}Ma^2 \\
I_{zz} &= \int (x^2 + y^2)\, dm = I_{xx} + I_{yy} = \frac{1}{3}M(a^2 + b^2) \\
I_{xz} &= -\int xz\, dm = 0 = I_{zx} = I_{yz} = I_{zy} \\
I_{xy} &= -\int xy\, dm = -\frac{M}{ab}\int_0^a dx\, x \int_0^b dy\, y = -\frac{M}{ab}\frac{a^2}{2}\frac{b^2}{2} = -\frac{1}{4}Mab = I_{yx}
\end{aligned}$$

$$\text{so} \qquad \underline{\underline{I}} = \frac{M}{12}\begin{bmatrix} 4b^2 & -3ab & 0 \\ -3ab & 4a^2 & 0 \\ 0 & 0 & 4(a^2 + b^2) \end{bmatrix}$$

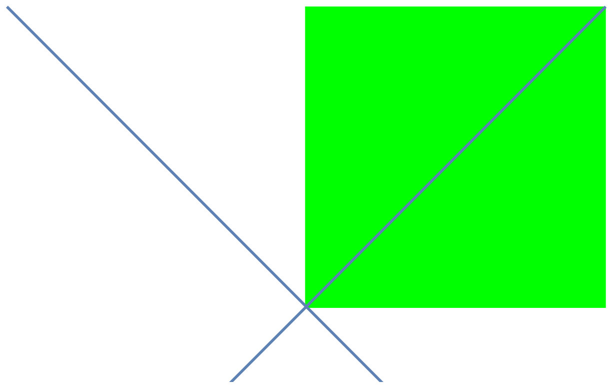
See the associated MATHEMATICA notebook for the rest. The eigenvalues are

$$\lambda_3 = 4\left(a^2 + b^2\right) \qquad \lambda_1 = 2a^2 - \sqrt{4a^4 + a^2b^2 + 4b^4} + 2b^2 \qquad \lambda_2 = 2a^2 + \sqrt{4a^4 + a^2b^2 + 4b^4} + 2b^2$$

and $\lambda_3 = \lambda_1 + \lambda_2$, as it should be for this flat plate. The eigenvectors are the rows of

$$\begin{bmatrix} 0 & 0 & 1 \\ -\frac{-2a^2 - \sqrt{4a^4 + a^2b^2 + 4b^4} + 2b^2}{3ab} & 1 & 0 \\ -\frac{-2a^2 + \sqrt{4a^4 + a^2b^2 + 4b^4} + 2b^2}{3ab} & 1 & 0 \end{bmatrix}$$

For the square I picked $a = b = 1$ and for the rectangle $a = 4$ and $b = 1$. The drawings are



In both cases, of course, the third principle axis points out of the page.

(3) As shown in Figure 10.9 (a), the angular velocity vector $\boldsymbol{\omega}$ rotates about the $\hat{\mathbf{e}}_3$ axis with angular velocity $\boldsymbol{\Omega}_b$ in the body frame. Since angular velocity vectors add as usual, the angular velocity $\boldsymbol{\Omega}_s$ must equal the sum of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}_b$, that is $\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_b + \boldsymbol{\omega}$. (I'm not so pleased with this explanation, but I think I see how the same result can be obtained in the body from, with appropriate sign changes of the vectors.)

We know from class that the three vectors $\hat{\mathbf{e}}_3$, $\boldsymbol{\omega}$, and $\boldsymbol{\Omega}_s$ (which is in the same direction as $\mathbf{L}$) all lie in a plane. If we evaluate $\boldsymbol{\Omega}_s = \boldsymbol{\Omega}_b + \boldsymbol{\omega}$ by components in the plane but perpendicular to $\hat{\mathbf{e}}_3$, then we write $(\boldsymbol{\Omega}_s)_\perp = (\boldsymbol{\omega})_\perp$. Referring to the angles in Figure 10.9,

$$(\boldsymbol{\Omega}_s)_\perp = \Omega_s \sin\theta \qquad \text{so} \qquad \Omega_s = \frac{1}{\sin\theta}(\boldsymbol{\omega})_\perp$$

Now $(\boldsymbol{\omega})_\perp$ is either the $\hat{\mathbf{e}}_1$ or $\hat{\mathbf{e}}_2$ component of $\boldsymbol{\omega}$, which are the same since the body is symmetric, and similarly for $(\mathbf{L})_\perp$. Therefore $(\mathbf{L})_\perp = L_1 = \lambda_1\omega_1 = \lambda_1(\boldsymbol{\omega})_\perp$. It is also clear from Figure 10.9 that $(\mathbf{L})_\perp = L\sin\theta$. Putting this all together gives

$$\Omega_s = \frac{1}{\sin\theta}\frac{(\mathbf{L})_\perp}{\lambda_1} = \frac{L}{\lambda_1}$$

(4) From the θ-equation Taylor (10.107) and $\ddot{\theta} = 0$ and $\dot{\phi} \equiv \Omega$. Then, from Taylor (10.99), we have $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$. Substituting for this expression and dividing out $\sin\theta$ gives

$$\lambda_1\Omega^2\cos\theta - \lambda_3\omega_3\Omega + MgR = 0$$

Solving for Ω requires $\lambda_3\omega_3 \pm (\lambda_3^2\omega_3^2 - 4MgR\lambda_1\cos\theta)^{1/2} \approx \lambda_3\omega_3 \pm \lambda_3\omega_3(1 - 2MgR\lambda_1\cos\theta/\lambda_3^2\omega_3^2)$, assuming that $\lambda_3^2\omega_3^2 = L_3^2 \gg MgR\lambda_1\cos\theta$. Thus, the two precession frequencies are

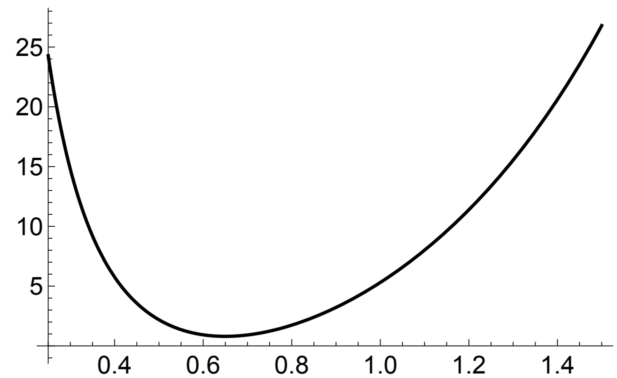
$$\Omega = \frac{\lambda_3\omega_3}{\lambda_1\cos\theta} \equiv \Omega_{\text{fast}} \qquad \text{and} \qquad \Omega = \frac{MgR}{\lambda_3\omega_3} \equiv \Omega_{\text{slow}}$$

So, "weak torque" means torque$= MgR\sin\theta \ll L_3^2\tan\theta/\lambda_1$. The second (slower) frequency is just what we derived for a symmetric spinning top under a "weak" gravitational torque.

See Taylor (10.83). As for the first (faster) frequency, note that $L_3 = L\cos\theta$ where L is the vertical, and constant, angular momentum component. Thus $\Omega_{\rm slow} = L/\lambda_1$ which is just the "space frame" precession rate of a freely rotating body. See Taylor (10.96) and Fig.10.9. Indeed, the gravitation torque does not enter here, so the "fixed point" plays no role.

(5) Since L_3 and λ_3 are constants, the second term is a constant, so it is unimportant when using energy to discuss the motion. Use MATHEMATICA to do the rest.

The plot at the right is for $L_z = 8$, $L_3 = 10$, and $\lambda_1 = 1 = MgR$. The effective potential is a minimum for $\theta = \theta_0 = 0.6496 \approx \cos^{-1}(8/10) = 0.6435$. From Taylor (10.104) $\dot{\phi} = (L_z - L_3\cos\theta_0)/\lambda_1\sin^2\theta_0 = 0.1008$. The approximate formula, for a top precessing under a "weak torque", Taylor (10.83) or (10.111), gives $\Omega = MgR/L_3 = 1/10$. That's very good agreement!



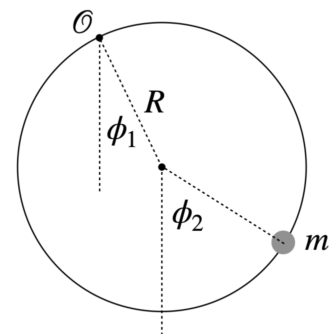
PHYS3101 Analytical Mechanics   Homework #8   Due 24 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) If $\underline{\underline{I}}^{\mathrm{CM}}$ is the moment of inertia tensor for some rigid body of mass M about its center of mass, and $\underline{\underline{I}}$ is the inertia tensor about a point displaced an amount $\mathbf{\Delta} = \hat{\mathbf{x}}\Delta_x + \hat{\mathbf{y}}\Delta_y + \hat{\mathbf{z}}\Delta_z$ from the center of mass, show that

$$\underline{\underline{I}}_{ij} = \underline{\underline{I}}_{ij}^{\mathrm{CM}} + M\left(\mathbf{\Delta}^2\delta_{ij} - \Delta_i\Delta_j\right)$$

(2) A hoop of mass m and radius R hanging from a fixed point $\mathcal{O}$ swings freely in the vertical plane. A bead, also of mass m, slides without friction on the hoop. Using the angles ϕ_1 and ϕ_2 as shown in the figure on the right as generalized coordinates, write the Lagrangian. (You will need to find the moment of inertia of the hoop about a point on the hoop. This is a very simple calculation if you use the results of Problem **(1)**.) Then for $\phi_1 \ll 1$ and $\phi_2 \ll 1$, determine the equations of motion and find the eigenfrequencies and describe the normal modes.



(3) Two equal masses m are connected by three springs and slide freely on a horizontal track. The outer springs are attached to fixed walls and have stiffness k, and the middle spring has stiffness εk. (Note that this ε is dimensionless, and is not the same as ϵ used in Taylor Section 11.3.) Derive expressions for the positions $x_1(t)$ and $x_2(t)$ of the two masses assuming they both start from rest and $x_1(0) = a$ and $x_2(0) = 0$. (You can carry out this calculation any way you like, including just solving the differential equations using Mathematica.) Make plots of $x_1(t)/a$ and $x_2(t)/a$ for $\epsilon = 1/10$ as a function of $\tau = \Omega t$ where $\Omega^2 = k/m$. Your result should look like Figure 11.8 in Taylor. Also plot the normal modes and show that they oscillate with two distinct, single frequencies.

(4) Consider a potential energy function $U(q_1, q_2, \ldots, q_n)$ where the q_i are n generalized coordinates that describe a system of N masses. Assume that when all of the $q_i = 0$, then the function U is at a local minimum. Find the Euler-Lagrange equations of motion for the case when the q_i do not move far from equilibrium, and show that the equations of motion can be written as

$$\underline{\underline{M}}\,\underline{\ddot{q}} + \underline{\underline{K}}\,\underline{q} = 0$$

where $\underline{\underline{M}}$ and $\underline{\underline{K}}$ are real, symmetric matrices. (This is most easily done by deriving expressions for $\underline{\underline{M}}$ and $\underline{\underline{K}}$ in terms of what you are given. See Taylor Section 7.8.)

(5) Consider a frictionless rigid horizontal hoop of radius R. Onto this hoop are threaded three beads with masses $2m$, m, and m, and, between the beads, three identical springs, each with force constant k. Solve for the three normal frequencies and find and describe the three normal modes.

(1) This follows directly from the definition of the inertia tensor. If $\mathbf{r}$ measures the position with respect to the center of mass, and $\mathbf{r}'$ with respect to an arbitrary point, then

$$\begin{aligned}
I_{ij} &= \int dm \left[\mathbf{r}'^2 \delta_{ij} - r_i' r_j'\right] \\
&= \int dm \left[(\mathbf{r}+\mathbf{\Delta})^2 \delta_{ij} - (r_i+\Delta_i)(r_j+\Delta_j)\right] \\
&= \int dm \left[(\mathbf{r}^2 + 2\mathbf{r}\cdot\mathbf{\Delta} + \mathbf{\Delta}^2)\delta_{ij} - (r_i r_j + r_i\Delta_j + r_j\Delta_i + \Delta_i\Delta_j)\right] \\
&= \int dm(\mathbf{r}^2\delta_{ij} - r_i r_j) + 2\left[\int dm\, \mathbf{r}\right]\cdot \mathbf{\Delta}\delta_{ij} - \left[\int dm\, r_i\right]\Delta_j - \left[\int dm\, r_j\right]\Delta_i \\
&\quad + \left[\int dm\right]\mathbf{\Delta}^2\delta_{ij} - \left[\int dm\right]\Delta_i\Delta_j = I_{ij}^{\rm CM} + 0 + 0 + 0 + M\mathbf{\Delta}^2\delta_{ij} - M\Delta_i\Delta_j \\
&= I_{ij}^{\rm CM} + M\left(\mathbf{\Delta}^2\delta_{ij} - \Delta_i\Delta_j\right)
\end{aligned}$$

where the integrals indicated equal zero because they are measuring the position of the center of mass in the center of mass frame. This is the generalization of the "parallel axis theorem" that you likely learned about in your first physics course.

(2) The kinetic energy for the bead is just what we derived in class for the double pendulum, but with $m_2 = m$ and $L_1 = L_2 = R$, so

$$T_{\rm bead} = \frac{1}{2}mR^2\dot\phi_1^2 + \frac{1}{2}mR^2\dot\phi_2^2 + mR^2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)$$

The kinetic energy of the hoop is $I\dot\phi_1^2/2$ where I is the moment of inertia about an axis perpendicular to the hoop's plane and passing through a point on the hoop. However the moment of inertia for a parallel axis passing through the center of the hoop is $I^{\rm CM} = mR^2$. If we let z measure the position perpendicular to the plane of the hoop, then we displace the axis by a vector $\mathbf{\Delta} = R\hat{\mathbf{x}}$, for example, so

$$I = I_{zz} = I_{zz}^{\rm CM} + mR^2 = 2mR^2$$

Therefore, the kinetic energy of the hoop is

$$T_{\rm hoop} = \frac{1}{2}I\dot\phi_1^2 = mR^2\dot\phi_1$$

The potential energy of the hoop and bead is also the same as for the double pendulum, since we concentrate the mass of the hoop at its center, so

$$U_{\rm hoop} = mgR(1-\cos\phi_1) \qquad \text{and} \qquad U_{\rm bead} = mgR(1-\cos\phi_1) + mgR(1-\cos\phi_2)$$

Gathering some terms, the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}(\phi_1,\phi_2,\dot\phi_1,\dot\phi_2) &= \frac{3}{2}mR^2\dot\phi_1^2 + \frac{1}{2}mR^2\dot\phi_2^2 + mR^2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) \\
&\quad -2mgR(1-\cos\phi_1) - mgR(1-\cos\phi_2)
\end{aligned}$$

Linearizing the Lagrangian for small angles, we get

$$\mathcal{L}(\phi_1,\phi_2,\dot\phi_1,\dot\phi_2) = \frac{3}{2}mR^2\dot\phi_1^2 + \frac{1}{2}mR^2\dot\phi_2^2 + mR^2\dot\phi_1\dot\phi_2 - mgR\phi_1^2 - \frac{1}{2}mgR\phi_2^2$$

The two Euler-Lagrange equations become

$$3mR^2\ddot\phi_1 + mR^2\ddot\phi_2 = -2mgR\phi_1 \qquad \text{and} \qquad mR^2\ddot\phi_2 + mR^2\ddot\phi_1 = -mgR\phi_2$$

Dividing through by mR^2 and defining $\omega_0^2 = g/L$, we write these equations as

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \ddot\phi_1 \\ \ddot\phi_2 \end{bmatrix} = -\omega_0^2\begin{bmatrix} 2\phi_1 \\ \phi_2 \end{bmatrix} = -\begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Using our nomenclature from class, this means that

$$\underline{\underline{M}} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \underline{\underline{K}} = \begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}$$

We get the eigenfrequencies ω from

$$\begin{aligned}
\left|\underline{\underline{K}} - \omega^2\underline{\underline{M}}\right| &= \begin{vmatrix} 2\omega_0^2 - 3\omega^2 & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix} \\
&= (2\omega_0^2 - 3\omega^2)(\omega_0^2 - \omega^2) - \omega^4 \\
&= 2\omega_0^4 - 5\omega_0^2\omega^2 + 2\omega^4 = (2\omega_0^2 - \omega^2)(\omega_0^2 - 2\omega^2) = 0
\end{aligned}$$

Therefore, the eigenfrequencies and eigenmodes are

$$\omega = \omega_0\sqrt{2} \qquad \text{so} \qquad \begin{bmatrix} -4\omega_0^2 & -2\omega_0^2 \\ -2\omega_0^2 & -\omega_0^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad a_2 = -2a_1$$

and the bead oscillates out of phase and with twice the amplitude of the hoop; and

$$\omega = \omega_0\frac{1}{\sqrt{2}} \qquad \text{so} \qquad \begin{bmatrix} \omega_0^2/2 & -\omega_0^2/2 \\ -\omega_0^2/2 & \omega_0^2/2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad a_2 = a_1$$

and the bead oscillates in phase and with the same amplitude as the hoop.

(3) There are different ways to approach this. I tried to follow Taylor Sec 11.3, but I had trouble using MATHEMATICA to do the complex arithmetic. So, I used instead the form with the $\pm\omega$ solutions with their own (real) coefficients. That is

$$\begin{aligned}
x_1(t) &= a_1e^{it\omega_1} + a_2e^{it\omega_2} + b_1e^{-it\omega_1} + b_2e^{-it\omega_2} \\
x_2(t) &= a_1e^{it\omega_1} - a_2e^{it\omega_2} + b_1e^{-it\omega_1} - b_2e^{-it\omega_2}
\end{aligned}$$

Changing to parameters given in the problem, we write

$$k_2 = \varepsilon k \qquad \omega_1 = \sqrt{\frac{k}{m}} = \Omega \qquad \omega_2 = \sqrt{\frac{k+2\varepsilon k}{m}} = \Omega\sqrt{1+2\varepsilon} \qquad \tau = \Omega t$$

(Note that our ε is not the same as the parameter ϵ in Taylor.) Inserting the initial conditions $x_1(0) = a$ and $x_2(0) = \dot x_1(0) = \dot x_2(0) = 0$ we solve for the coefficients and find

$$a_1 = b_1 = a_2 = b_2 = \frac{a}{4}$$

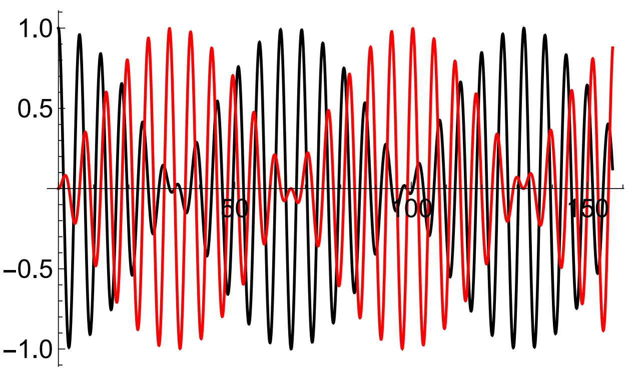
The expressions we want to plot become

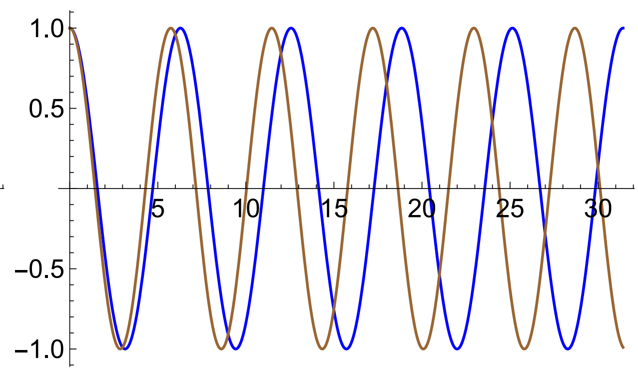
$$\frac{1}{a}x_1(t) = \frac{\cos(\tau)}{2} + \frac{1}{2}\cos\left(\sqrt{\frac{6}{5}}\tau\right) \qquad \text{and} \qquad \frac{1}{a}x_2(t) = \frac{\cos(\tau)}{2} - \frac{1}{2}\cos\left(\sqrt{\frac{6}{5}}\tau\right)$$

and the normal modes are

$$\xi_1(t) = x_1(t) + x_2(t) = \cos(\tau) \qquad \text{and} \qquad \xi_2(t) = x_1(t) - x_2(t) = \cos\left(\sqrt{\frac{6}{5}}\tau\right)$$

which are indeed single frequency. Their plots are





Indeed, the plots of $x_1(t)$ and $x_2(t)$ look like Figure 11.8 in Taylor.

(4) The kinetic energy of N particles was derived in Section 7.8 in Taylor, namely

$$\begin{aligned} T &= \sum_\alpha \frac{1}{2} m_\alpha \mathbf{r}_\alpha^2 = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k \\ \text{where} \qquad M_{jk} &= M_{ij}(q_1, q_2, \ldots, q_n) = \sum_\alpha m_\alpha \left(\frac{\partial \mathbf{r}_\alpha}{\partial q_j}\right)\cdot\left(\frac{\partial \mathbf{r}_\alpha}{\partial q_k}\right) \end{aligned}$$

for n generalized coordinates. This is exactly as written in Taylor, except that I'm using the notation M_{ij} instead of A_{ij}. We can write the potential energy function as a multi-variable Taylor expansion as

$$U(q_1, q_2, \ldots, q_n) = U_0 + \sum_i \frac{\partial U}{\partial q_i} q_i + \frac{1}{2}\sum_{j,k} \frac{\partial^2 U}{\partial q_j \partial q_k} q_j q_k + \cdots$$

where U_0 is the constant energy at the equilibrium point, and which can be ignored when we write the Lagrangian. Also, since we are evaluating at a local minimum, all of the first derivatives will be zero. Furthermore, we could in principle expand the M_{jk}, but in this case the lowest order is all that matters, so we consider M_{jk} a constant. The Lagrangian is therefore

$$\mathcal{L}(\underline{q}, \underline{\dot{q}}) = \frac{1}{2}\sum_{j,k}\left[M_{jk}\dot{q}_j\dot{q}_k - K_{jk}q_jq_k\right] \qquad \text{where} \qquad K_{jk} \equiv \frac{\partial^2 U}{\partial q_j \partial q_k}$$

and the M_{jk} and K_{jk} are understood to be constants, evaluated at the equilibrium point.

It is plain to see that $M_{jk} = M_{kj}$ and $K_{jk} = K_{kj}$. That is, these are both symmetric matrices.

The Euler-Lagrange equation for coordinate q_i is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} &= \frac{1}{2}\sum_{j,k} M_{jk}(\delta_{ij}\ddot{q}_k + \delta_{ik}\ddot{q}_j) + \frac{1}{2}\sum_{j,k} K_{jk}(\delta_{ij}q_k + \delta_{ik}q_j) \\
&= \frac{1}{2}\sum_{k} M_{ik}\ddot{q}_k + \frac{1}{2}\sum_{j} M_{ji}\ddot{q}_j + \frac{1}{2}\sum_{k} K_{ik}q_k + \frac{1}{2}\sum_{j} K_{ji}q_j \\
&= \frac{1}{2}\sum_{j} M_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j} M_{ij}\ddot{q}_j + \frac{1}{2}\sum_{j} K_{ij}q_j + \frac{1}{2}\sum_{j} K_{ij}q_j \\
&= \sum_{j} M_{ij}\ddot{q}_j + \sum_{j} K_{ij}q_j = 0
\end{aligned}$$

where the third line makes use of the symmetry of the matrices, and also uses the dummy index switch $k \to j$. Written in terms of matrices, this final equation becomes

$$\underline{\underline{M}}\,\underline{\ddot{q}} + \underline{\underline{K}}\,\underline{q} = 0$$

(5) Let the beads be labeled $m_1 = 2m$, $m_2 = m$, and $m_3 = m$, with ϕ_1, ϕ_2, and ϕ_3 as the angles that locate the masses round the hoop. The Lagrangian is therefore

$$\mathcal{L} = mR^2\dot{\phi}_1^2 + \frac{1}{2}mR^2\dot{\phi}_2^2 + \frac{1}{2}R^2\dot{\phi}_3^2 - \frac{1}{2}kR^2(\phi_2 - \phi_1)^2 - \frac{1}{2}kR^2(\phi_3 - \phi_2)^2 - \frac{1}{2}kR^2(\phi_1 - \phi_3)^2$$

The equations of motion are

$$\begin{aligned}
2mR^2\ddot{\phi}_1 &= -kR^2(\phi_2 - \phi_1)(-1) - kR^2(\phi_1 - \phi_3) = -2kR^2\phi_1 + kR^2\phi_2 + kR^2\phi_3 \\
mR^2\ddot{\phi}_2 &= -kR^2(\phi_2 - \phi_1) - kR^2(\phi_3 - \phi_2)(-1) = kR^2\phi_1 - 2kR^2\phi_2 + kR^2\phi_3 \\
mR^2\ddot{\phi}_3 &= -kR^2(\phi_3 - \phi_2) - kR^2(\phi_1 - \phi_3)(-1) = kR^2\phi_1 + kR^2\phi_2 - 2kR^2\phi_3
\end{aligned}$$

Dividing through by mR^2 and defining $\omega_0^2 = k/m$, this is written as $\underline{\underline{M}}\,\underline{\ddot{\phi}} = -\underline{\underline{K}}\,\underline{\phi}$ where

$$\underline{\underline{M}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \underline{\underline{K}} = \begin{bmatrix} 2\omega_0^2 & -\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega_0^2 & 2\omega_0^2 \end{bmatrix}$$

Use Mathematica to find the determinant as

$$\left|\underline{\underline{K}} - \omega^2\underline{\underline{M}}\right| = 2\omega^2\left(\omega^2 - 3\omega_0^2\right)\left(\omega^2 - 2\omega_0^2\right) = 0$$

so the eigenfrequencies are $\omega_1^2 = 0$, $\omega_2^2 = 3\omega_0^2$, and $\omega_3^2 = 2\omega_0^2$.

For $\omega^2 = 0$, we find (using the Mathematica notebook)

$$\frac{1}{\omega_0^2}(\underline{\underline{K}} - \omega^2\underline{\underline{M}})\underline{a} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Each of the three equations says that one of the a is equal to the sum of the other two, that is $a_1 = a_2 = a_3$. In other words, the mode with zero frequency just means that the beads move around the hoop in unison. This makes perfect sense.

For $\omega^2 = 3\omega_0^2$, we find (using the Mathematica notebook)

$$\frac{1}{\omega_0^2}(\underline{\underline{K}} - \omega^2\underline{\underline{M}})\underline{a} = \begin{bmatrix} -4 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Subtracting the first equation from the second gives $4a_1 = 0$, that is $a_1 = 0$ and the first $(2m)$ mass does not move. The second or third equations give $a_2 = -a_3$. In other words, the heavy mass stays put and the two lighter masses on either side opposite with the same amplitudes and 180° out of phase with each other.

For $\omega^2 = 2\omega_0^2$, we find (using the Mathematica notebook)

$$\frac{1}{\omega_0^2}(\underline{\underline{K}} - \omega^2\underline{\underline{M}})\underline{a} = \begin{bmatrix} -2 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The third equation says $a_2 = -a_1$ and the second equation says $a_3 = -a_1$. In other words, all three masses oscillate with the same amplitude, but the heavy mass oscillates 180° out of phase with the light ones.

PHYS3101 Analytical Mechanics   Homework #9   Due 31 Oct 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) We solved in class the problem of the double pendulum. (See also Taylor Section 11.4.) Specialize to the case of equal masses and equal lengths. Use this solution to write the (two-dimensional) vector $\underline{\phi}(t)$ as a sum over the normal mode vectors with coefficients $\xi_i(t)$. Turn this around to determine the $\xi_i(t)$ in terms of $\phi_1(t)$ and $\phi_2(t)$, and then show that the $\xi_i(t)$ oscillate with the eigenfrequencies ω_i by deriving the differential equations for the $\xi_i(t)$ from those for $\phi_1(t)$ and $\phi_2(t)$.

(2) A mass m moves in one horizontal direction x on a frictionless track. The mass is connected to a spring with stiffness k, which is itself attached to a fixed wall. (So far, this is a very familiar, simple problem.) Now assume that the spring is not massless, but has a total mass μ, uniformly distributed along its length, even as it stretches. Find the Hamiltonian $\mathcal{H}(x,p)$ and solve Hamilton's equations to find the oscillation frequency ω in terms of m, μ, and k. (Remember that the spring is continuous and a small piece of it that is close to the wall is moving more slowly than a small piece that is closer to the mass.)

(3) A bead of mass m moves without friction along a curved wire that lies entirely in the vertical plane. The shape of the wire is given by the function $y = h(x)$. Find the Hamiltonian $\mathcal{H}(x,p)$ and show that Hamilton's equations give the result is the same as Newton's Second Law in terms of a position variable s that measures distance along the wire. (My apologies for the somewhat messy algebra and calculus.)

(4) In class, we used the Lagrangian approach to solve the problem of a bead of mass m constrained to a circular wire hoop of radius R which itself rotated about a vertical axis with angular velocity ω. (See also Taylor Example. 7.6, with Figure 7.9.) Construct the Hamiltonian $\mathcal{H}(\theta, p_\theta)$ and comment on what is peculiar about it. Then show that Hamilton's equations lead to the same differential equation for $\ddot{\theta}$, namely Taylor (7.69).

(5) In class, we showed that the "potential energy" term in a Lagrangian for a particle with charge q moving in a region of magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ is given by $q\dot{\mathbf{r}} \cdot \mathbf{A}$. (See also Taylor Section 7.9. I will stick with SI units here for the electromagnetic quantities.) Use this to find the Hamiltonian $\mathcal{H}(\mathbf{r}, \mathbf{p})$ for a particle with charge q and mass m in a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E} = -\boldsymbol{\nabla} V$, and show that Hamilton's equations reduce to the Lorentz force law, namely $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$, where $\mathbf{v}$ is the velocity vector of the mass.

PHYS3101 Analytical Mechanics Homework #9 Solutions

(1) The normal modes for the double pendulum are given by

$$\begin{aligned} \omega_1^2 &= (2-\sqrt{2})\omega_0^2 \qquad \text{with} \qquad \underline{a}_1 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \\ \text{and} \qquad \omega_2^2 &= (2+\sqrt{2})\omega_0^2 \qquad \text{with} \qquad \underline{a}_2 = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \end{aligned}$$

Therefore, the expansion in terms of normal modes is

$$\underline{\phi}(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \sum_{i=1}^{2} \xi_i(t)\underline{a}_i = \begin{bmatrix} \xi_1(t)+\xi_2(t) \\ \sqrt{2}\xi_1(t) - \sqrt{2}\xi_2(t) \end{bmatrix}$$

This is easily solved for $\xi_1(t)$ and $\xi_2(t)$, namely

$$\begin{aligned} \xi_1(t) &= \frac{1}{2\sqrt{2}}\left[\sqrt{2}\phi_1(t)+\phi_2(t)\right] \\ \xi_2(t) &= \frac{1}{2\sqrt{2}}\left[\sqrt{2}\phi_1(t)-\phi_2(t)\right] \end{aligned}$$

The equations of motion for $\phi_1(t)$ and $\phi_2(t)$ are given by Taylor (11.41) and (11.42). For equal masses m and equal lengths L, and defining $\omega_0^2 = g/L$, these become

$$\begin{aligned} 2\ddot{\phi}_1 + \ddot{\phi}_2 &= -2\omega_0^2\phi_1 \\ \ddot{\phi}_1 + \ddot{\phi}_2 &= -\omega_0^2\phi_2 \end{aligned}$$

which is equivalent to (11.43) with (11.45). Solving for $\ddot{\phi}_1$ and $\ddot{\phi}_2$ we get

$$\begin{aligned} \ddot{\phi}_1 &= -2\omega_0^2\phi_1 + \omega_0^2\phi_2 \\ \ddot{\phi}_2 &= +2\omega_0^2\phi_1 - 2\omega_0^2\phi_2 \end{aligned}$$

This makes it straightforward to determine the differential equations for $\xi_1(t)$ and $\xi_2(t)$.

$$\begin{aligned} \ddot{\xi}_1(t) &= \frac{1}{2\sqrt{2}}\left[\sqrt{2}\ddot{\phi}_1(t)+\ddot{\phi}_2(t)\right] = \frac{1}{2\sqrt{2}}\left[\sqrt{2}(-2\omega_0^2\phi_1+\omega_0^2\phi_2)+(2\omega_0^2\phi_1-2\omega_0^2\phi_2)\right] \\ &= \frac{\omega_0^2}{2\sqrt{2}}\left[(-2\sqrt{2}+2)\phi_1+(\sqrt{2}-2)\phi_2\right] = \frac{\omega_0^2}{2\sqrt{2}}\left[(-2+\sqrt{2})\sqrt{2}\phi_1+(\sqrt{2}-2)\phi_2\right] \\ &= -(2-\sqrt{2})\omega_0^2\,\frac{1}{2\sqrt{2}}\left[\sqrt{2}\phi_1+\phi_2\right] = -\omega_1^2\xi_1(t) \\ \ddot{\xi}_2(t) &= \frac{1}{2\sqrt{2}}\left[\sqrt{2}\ddot{\phi}_1(t)-\ddot{\phi}_2(t)\right] = \frac{1}{2\sqrt{2}}\left[\sqrt{2}(-2\omega_0^2\phi_1+\omega_0^2\phi_2)-(2\omega_0^2\phi_1-2\omega_0^2\phi_2)\right] \\ &= \frac{\omega_0^2}{2\sqrt{2}}\left[(-2\sqrt{2}-2)\phi_1+(\sqrt{2}+2)\phi_2\right] = \frac{\omega_0^2}{2\sqrt{2}}\left[(-2-\sqrt{2})\sqrt{2}\phi_1+(\sqrt{2}+2)\phi_2\right] \\ &= -(2+\sqrt{2})\omega_0^2\,\frac{1}{2\sqrt{2}}\left[\sqrt{2}\phi_1-\phi_2\right] = -\omega_2^2\xi_2(t) \end{aligned}$$

The differential equations for $\xi_1(t)$ and $\xi_2(t)$ demonstrate that these coordinates in fact oscillate solely with their eigenfrequencies.

(2) Calculating the kinetic energy of the spring takes some thought and a little calculus. If the spring has length ℓ, and we let z measure position along the spring, then the speed of small piece of length dz would be $\dot{z} = (z/\ell)\dot{x}$ where $x(t)$ is the position of the end of the spring attached to the mass. Therefore, the kinetic energy of the spring is

$$T_{\text{spring}} = \int_0^\ell \frac{1}{2}\frac{\mu}{\ell} dz\, \dot{z}^2 = \frac{1}{2}\frac{\mu}{\ell^3}\left[\int_0^\ell z^2\, dz\right]\dot{x}^2 = \frac{1}{6}\mu\dot{x}^2$$

The Lagrangian is therefore

$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{6}\mu\left(\frac{\dot{x}}{2}\right)^2 - \frac{1}{2}kx^2 = \frac{1}{2}\left(m + \frac{1}{3}\mu\right)\dot{x}^2 - \frac{1}{2}kx^2$$

The conjugate momentum is

$$p = \frac{\partial\mathcal{L}}{\partial\dot{x}} = \left(m + \frac{1}{3}\mu\right)\dot{x} \qquad \text{so} \qquad \dot{x} = \frac{p}{m + \mu/3}$$

The Hamiltonian is therefore

$$\mathcal{H}(x,p) = p\dot{x} - \mathcal{L} = \frac{p^2}{m+\mu/3} - \frac{1}{2}\frac{p^2}{m+\mu/3} + \frac{1}{2}kx^2 = \frac{1}{2}\frac{3p^2}{3m+\mu} + \frac{1}{2}kx^2$$

Hamilton's equations give

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p} = \frac{3p}{3m+\mu} \qquad \text{and} \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial x} = -kx$$

Therefore

$$\frac{3m+\mu}{3}\ddot{x} = -kx \qquad \text{so} \qquad \ddot{x} = -\omega^2 x \qquad \text{where} \qquad \omega^2 = \frac{3k}{3m+\mu}$$

This agrees with the statement of problem 13.6 in Taylor.

(3) The potential energy is simply $U = mgh(x)$. The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[h'(x)\right]^2\dot{x}^2 = \frac{1}{2}m\left[1 + h'^2(x)\right]\dot{x}^2$$

We might as well just realize that the Hamiltonian is the energy for this problem, so

$$p = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\left[1 + h'^2(x)\right]\dot{x} \qquad \text{so} \qquad \mathcal{H}(x,p) = \frac{p^2}{2m}\frac{1}{1+h'^2(x)} + mgh(x)$$

Hamilton's equations therefore give

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p} = \frac{p}{m}\frac{1}{1+h'^2(x)} \qquad \text{and} \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial x} = \frac{p^2}{m}\frac{h'h''}{\left[1+h'^2\right]^2} - mgh'(x)$$

We are going to want to compare this to "$F = m\ddot{s}$" at some point, so let's eliminate momentum from the equations. The first equation gives us

$$\dot{p} = m\ddot{x}(1 + h'^2) + 2m\dot{x}^2 h'h''$$

Inserting the first of Hamilton's equations into the second, we get

$$\dot{p} = m\dot{x}^2 h' h'' - mgh'$$

so Hamilton's equations are reduced to the second order differential equation

$$\ddot{x}(1 + h'^2) + \dot{x}^2 h' h'' = -gh'$$

Now to write all this in terms of Newton's second law, we need to know the force F_s in the s direction, so

$$F_s = -\frac{dU}{ds} = -\frac{dU}{dy}\frac{dy}{ds} = -mg\frac{dy}{\sqrt{dx^2 + dy^2}} = -mg\frac{dy/dx}{\sqrt{1 + (dy/dx)^2}} = -mg\frac{h'}{\sqrt{1 + h'^2}}$$

Therefore

$$\begin{aligned}
m\ddot{s} = m\frac{d}{dt}\frac{ds}{dt} &= F_s = -mg\frac{h'}{\sqrt{1+h'^2}} \\
\frac{d}{dt}\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{d}{dt}\sqrt{\dot{x}^2(1 + h'^2)} &= -g\frac{h'}{\sqrt{1+h'^2}} \\
\frac{1}{2}\frac{2\dot{x}\ddot{x}(1 + h'^2) + 2\dot{x}^3 h' h''}{\sqrt{\dot{x}^2(1 + h'^2)}} &= -g\frac{h'}{\sqrt{1+h'^2}} \\
\text{or, finally,} \qquad \ddot{x}(1 + h'^2) + \dot{x}^2 h' h'' &= -gh'
\end{aligned}$$

which is the same result that we had with Hamilton's equations.

(4) We can start with the Lagrangian given in Taylor (7.68), namely

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta - mgR(1 - \cos\theta)$$

The momentum conjugate to θ is

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\dot{\theta}$$

so the Hamiltonian is

$$\begin{aligned}
\mathcal{H}(\theta, p_\theta) &= p_\theta\dot{\theta} - \mathcal{L} = \frac{p_\theta^2}{mR^2} - \frac{1}{2}\frac{p_\theta^2}{mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR(1 - \cos\theta) \\
&= \frac{1}{2}\frac{p_\theta^2}{mR^2} - \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR(1 - \cos\theta)
\end{aligned}$$

Notice that this is a case where the Hamiltonian *is not* equal to the total energy. (There is energy put in and taken out over time in order to keep the hoop spinning at a constant ω.) Now

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \qquad \text{so} \qquad \dot{p}_\theta = \ddot{\theta}mR^2$$

However, we also have

$$\dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$

Equating these two expressions for $\dot{p}_\theta$ and dividing through by mR^2, we get

$$\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right)$$

which is the same as Taylor (7.69).

(5) The Lagrangian is given by Taylor (7.103), namely

$$\begin{aligned}\mathcal{L}(\mathbf{r},\dot{\mathbf{r}}) &= \frac{1}{2}m\dot{\mathbf{r}}^2 - qV(\mathbf{r}) + q\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r}) \\ &= \frac{1}{2}m\dot{r}_i\dot{r}_i - qV(\mathbf{r}) + q\dot{r}_iA_i(\mathbf{r})\end{aligned}$$

where the sum over i is, of course, implied. The conjugate momenta are therefore

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{r}_i} = m\dot{r}_i + qA_i \qquad \text{so} \qquad \dot{\mathbf{r}} = \frac{1}{m}(\mathbf{p} - q\mathbf{A})$$

The Hamiltonian is

$$\begin{aligned}\mathcal{H}(\mathbf{r},\mathbf{p}) &= \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L} \\ &= \frac{1}{m}\mathbf{p}\cdot(\mathbf{p}-q\mathbf{A}) - \frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + qV(\mathbf{r}) - \frac{q}{m}(\mathbf{p}-q\mathbf{A})\cdot\mathbf{A} \\ &= \frac{1}{m}(\mathbf{p}-q\mathbf{A})\cdot(\mathbf{p}-q\mathbf{A}) - \frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + qV \\ &= \frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + qV = \frac{1}{2m}(p_j - qA_j)(p_j - qA_j) + qV\end{aligned}$$

Hamilton's equation for the time derivative of the ith coordinate is

$$\dot{r}_i = \frac{\partial\mathcal{H}}{\partial p_i} = \frac{1}{m}(p_i - qA_i) \qquad \text{so} \qquad \dot{p}_i = m\ddot{r}_i + q\frac{\partial A_i}{\partial r_j}\dot{r}_j$$

where the sum over j is, of course, implied. The other of Hamilton's equations is

$$\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial r_i} = -\frac{1}{m}(p_j - qA_j)\left(-q\frac{\partial A_j}{\partial r_i}\right) - q\frac{\partial V}{\partial r_i} = \dot{r}_j\,q\frac{\partial A_j}{\partial r_i} - q\frac{\partial V}{\partial r_i}$$

Equating the two expressions for $\dot{p}_i$ gives

$$m\ddot{r}_i = -q\frac{\partial V}{\partial r_i} + q\left(\dot{r}_j\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\dot{r}_j\right)$$

The left side is just the ith component of $m\ddot{\mathbf{r}}$. The first term on the right is the ith component of $-q\boldsymbol{\nabla}V = q\mathbf{E}$. For the term in parenthesis, write out the sums to get

$$\begin{aligned}\dot{r}_j\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\dot{r}_j &= \dot{x}\frac{\partial A_x}{\partial r_i} + \dot{y}\frac{\partial A_y}{\partial r_i} + \dot{z}\frac{\partial A_z}{\partial r_i} - \dot{x}\frac{\partial A_i}{\partial x} - \dot{y}\frac{\partial A_i}{\partial y} - \dot{z}\frac{\partial A_i}{\partial z} \\ &= \dot{x}\left(\frac{\partial A_x}{\partial r_i} - \frac{\partial A_i}{\partial x}\right) + \dot{y}\left(\frac{\partial A_y}{\partial r_i} - \frac{\partial A_i}{\partial y}\right) + \dot{z}\left(\frac{\partial A_z}{\partial r_i} - \frac{\partial A_i}{\partial z}\right)\end{aligned}$$

I'd like to find a slicker way to continue from here, but it's not coming to me. So, just consider $i = x$. In this case, we have

$$\begin{aligned}
m\ddot{x} &= qE_x + q\left[\dot{x}\left(\frac{\partial A_x}{\partial x} - \frac{\partial A_x}{\partial x}\right) + \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right] \\
&= qE_x + q\left[\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right] \\
&= qE_x + q\left[\dot{y}\left(\boldsymbol{\nabla} \times \mathbf{A}\right)_z - \dot{z}\left(\boldsymbol{\nabla} \times \mathbf{A}\right)_y\right] \\
&= qE_x + q\left[\dot{y}B_z - \dot{z}B_y\right] \\
&= qE_x + q(\dot{\mathbf{r}} \times \mathbf{B})_x
\end{aligned}$$

The other components will work out the same way, so this proves that Hamilton's equations are the same as the Lorentz force law.

PHYS3101 Analytical Mechanics Homework #10 Due 7 Nov 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Consider any two functions of generalized coordinates and momenta $f(\underline{q}, \underline{p})$ and $g(\underline{q}, \underline{p})$. Show each of the following for the Poisson Bracket, which is defined as

$$[f, g] \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

(a) $[g, f] = -[f, g]$
(b) $[q_i, q_j] = 0 = [p_i, p_j]$ but $[q_i, p_j] = \delta_{ij}$
(c) $\dot{q}_i = [q_i, \mathcal{H}]$ and $\dot{p}_i = [p_i, \mathcal{H}]$

If you've studied some Quantum Mechanics, does any of this look familiar to you?

(2) Write down the Hamiltonian for a simple plane pendulum of length ℓ with a bob of mass m. Use ϕ for the angle of the bob measured with respect to the vertical, as usual with $\phi = 0$ for the mass at its lowest point, and define $\omega^2 = g/\ell$.

(a) Find Hamilton's equations for ϕ and its conjugate momentum p. These equations become particularly simple of you write them in terms of dimensionless $\tilde{p} = p/m\ell^2\omega$.
(b) Plot $\phi(t)$, $p(t)$, and also the orbit (p versus ϕ) for the three sets of initial conditions (i) $\phi(0) = 0.1$ and $p(0) = 0$; (ii) $\phi(0) = 0.99\pi$ and $p(0) = 0$; and (iii) $\phi(0) = 0.99\pi$ and $p(0) = -0.05$. Briefly explain the motion these each describe.

(3) A beam of particles is moving along an accelerator pipe in the z-direction. The particles are uniformly distributed in a cylindrical volume of length L_0 (in the z-direction) and radius R_0. The particles have momenta uniformly distributed with p_z in an interval $p_0 \pm \Delta p$ and the transverse momentum $p_\perp$ inside a circle of radius $\Delta p_\perp$. To increase the particles' spatial density, the beam is focused by electric and magnetic fields, so that the radius shrinks to a smaller value R. What does Liouville's theorem tell you about the spread in the transverse momentum $p_\perp$ and the subsequent behavior of the radius R? (Assume that the focusing does not affect either L_0 or Δp_z.) Google "stochastic cooling" to learn why this is important.

(4) Suppose that you believe that antiprotons $\bar{p}$ exist, and you want to build an accelerator that would be able to produce them using the reaction $p + p \to p + p + p + \bar{p}$ where an incident beam proton of kinetic energy T is incident on a stationary target proton. What is the minimum amount of energy you need to produce antiprotons in this reaction? Can you find the name and location of the accelerator that was built to carry out this experiment?

(5) A particle of mass m has initial kinetic energy $T \gg mc^2$ and scatters from a stationary, identical particle. If the scattering is at 90° in the center of mass frame, find the outgoing opening angle between the two particles in the laboratory frame. You can work this out easily enough just by using conservation of energy and momentum, but a slick solution is to equate two inner products in the lab and CM frames. Be careful of how you use $T \gg mc^2$.

(1) The first part is trivial, that is

$$[g,f] = \sum_i \left(\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} \right) = -\sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) = -[f,g]$$

The second part is also simple. Just insert the appropriate quantities, and remember that the q_i and p_i are the independent variables. Also be careful to change the summation index.

$$\begin{aligned}
[q_i,q_j] &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k} \right) = \sum_k (\delta_{ik}\, 0 - 0\, \delta_{jk}) = 0 \\
[p_i,p_j] &= \sum_k \left(\frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right) = \sum_k (0\, \delta_{jk} - \delta_{ik}\, 0) = 0 \\
[q_i,p_j] &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k} \right) = \sum_k (\delta_{ik}\delta_{jk} - 0\, 0) = \delta_{ij}
\end{aligned}$$

The third part is similar, but you now invoke Hamilton's equations.

$$\begin{aligned}
[q_i,\mathcal{H}] &= \sum_j \left(\frac{\partial q_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} \right) = \sum_j \left(\delta_{ij}\frac{\partial \mathcal{H}}{\partial p_j} - 0\,\frac{\partial \mathcal{H}}{\partial q_j} \right) = \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \\
[p_i,\mathcal{H}] &= \sum_j \left(\frac{\partial p_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} \right) = \sum_j \left(0\,\frac{\partial \mathcal{H}}{\partial p_j} - \delta_{ij}\frac{\partial \mathcal{H}}{\partial q_j} \right) = -\frac{\partial \mathcal{H}}{\partial q_i} = \dot{p}_i
\end{aligned}$$

In quantum mechanics, you have very similar relationships for operators A and B that correspond to observables with the "commutator" $[A,B]$. In fact, you can show that the limit as $\hbar \to 0$ of $[A,B]/(i\hbar)$ is the Poisson bracket. This is a nice way to show how classical physics follows from quantum mechanics in this limit.

(2) Going through the steps from the Lagrangian, we have

$$\begin{aligned}
\mathcal{L}(\phi,\dot{\phi}) &= \frac{1}{2}m\ell^2\dot{\phi}^2 - mg\ell(1-\cos\phi) \\
p &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\ell^2\dot{\phi} \qquad \text{so} \qquad \dot{\phi} = \frac{p}{m\ell^2} \\
\mathcal{H}(\phi,p) &= p\dot{\phi} - \mathcal{L} = \frac{p^2}{m\ell^2} - \frac{1}{2}m\ell^2\left(\frac{p}{m\ell^2}\right)^2 + mg\ell(1-\cos\phi) \\
&= \frac{p^2}{2m\ell^2} + mg\ell(1-\cos\phi)
\end{aligned}$$

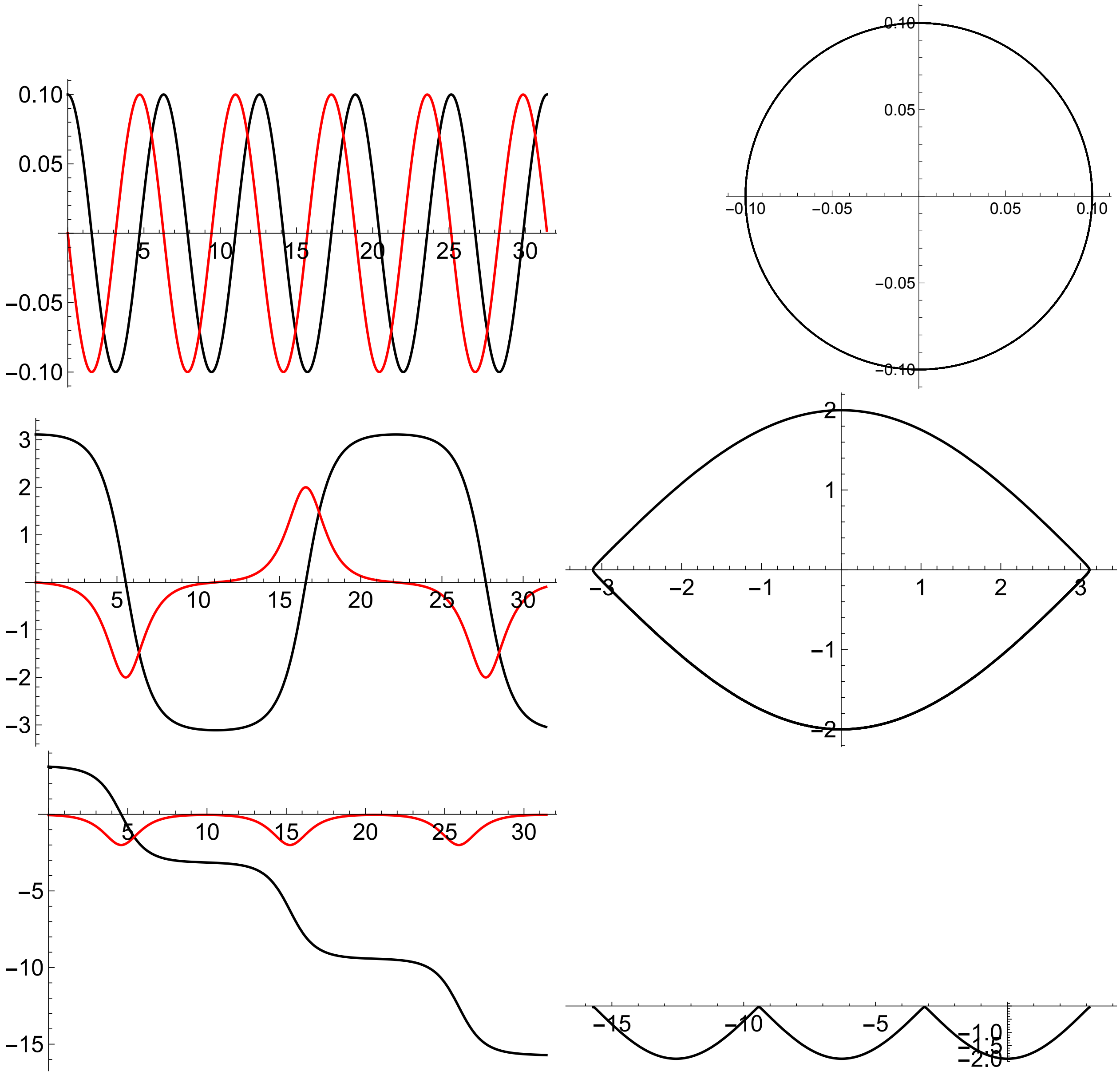
Therefore, Hamilton's equations are

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m\ell^2} \qquad \text{and} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial \phi} = -mg\ell\sin\phi$$

Written in terms of dimensionless $\tilde{p} = p/m\ell^2\omega$, these become

$$\dot{\phi} = \omega\tilde{p} \qquad \text{and} \qquad \dot{\tilde{p}} = -\omega\sin\phi$$

so solving the equations numerically as a function of $\tau = \omega t$ is straightforward. See the accompanying MATHEMATICA notebook. The plots are



The first set looks like a simple harmonic oscillator, since the angle never gets very large. The second is distorted because the angle does get large, so the equation is very nonlinear, but it is still closed. The third starts out a big angle with a negative kick, so the pendulum just keeps swinging around.

(3) With the lengths in z for the positions and for the momenta both fixed, the phase space volume must be proportional to $R^2 \Delta p_\perp^2$. This volume must remain fixed, so as R decreases, $\Delta p_\perp$ must increase, and this will lead to a "beam blowup" after R reaches some minimum value. This effect needed to be overcome in order to discover the $W^\pm$ and Z^0 bosons in $p]barp$ collisions at CERN, leading to the Nobel Prize-winning invention of *stochastic cooling* to dramatically reduce $\Delta p_\perp$ in the antiproton beam.

(4) You need to create enough energy so that you can make a $p\bar{p}$ pair and give them the necessary kinetic energy so that momentum is also conserved. This is an easy calculation in the center of mass frame where you only need the energy $2mc^2$ (where m is the mass of the proton or antiproton) plus the mass of the two initial protons, because the total momentum is zero. Therefore, we use four-momentum conservation and the invariance of the four-momentum squared to write

$$\begin{aligned}
(p_1+p_2)^2_{\text{CM}} &= (E_1+E_2)^2_{\text{CM}} - (\mathbf{p}_1+\mathbf{p}_2)^2_{\text{CM}}\,c^2 = (E_1+E_2)^2_{\text{CM}} = (4mc^2)^2 \\
(p_1+p_2)^2_{\text{Lab}} &= (E+mc^2)^2 - \mathbf{p}^2c^2 = (E+mc^2)^2 - (E^2-m^2c^4) = 2E(mc^2)+2(mc^2)^2 \\
(p_1+p_2)^2_{\text{Lab}} &= (p_1+p_2)^2_{\text{CM}} \qquad \text{so} \\
2E+2mc^2 &= 2T+4mc^2 = 16mc^2 \qquad \text{and} \\
T &= 6mc^2 = 6\times(0.9383\text{ GeV}) = 5.63\text{ GeV}
\end{aligned}$$

The accelerator was the Bevatron at Lawrence Berkeley Laboratory:
https://en.wikipedia.org/wiki/Bevatron

(5) For the slick solution, write $(p\cdot p')_{\text{Lab}} = (p\cdot p')_{\text{CM}}$ and $(p\cdot p_0)_{\text{Lab}} = (p\cdot p_0)_{\text{CM}}$ where p, p', and p_0 are the incident, scattered, and target particle four momenta, respectively. Then

$$(p\cdot p')_{\text{Lab}} = E\frac{E}{2} - E\frac{E}{2}\cos\theta = (p\cdot p')_{\text{CM}} = \left(\frac{E_{\text{CM}}}{2}\right)^2$$

where for $\theta_{\text{CM}} = 90^\circ$ we know that each of the outgoing particles has the same energy and momentum. Now $(p\cdot p_0)_{\text{CM}} = 2(E_{\text{CM}}/2)^2$ and $(p\cdot p_0)_{\text{Lab}} = Emc^2$, so we have

$$\frac{1}{2}Emc^2 = \frac{E^2}{2}(1-\cos\theta) \approx \frac{E^2}{2}\frac{\theta^2}{2} \qquad \text{giving} \qquad \theta = \left(\frac{2mc^2}{E}\right)^{1/2}$$

The straightforward approach is to just use conservation of energy and momentum. Let E', p', and p_L refer to the total energy, momentum, and longitudinal momentum of each of the two outgoing particles. Then, being careful with $T \gg mc^2$,

$$\begin{aligned}
E' &= \frac{1}{2}(E+mc^2) \qquad \text{so} \qquad E'^2 = \frac{1}{4}\left(E^2+2mc^2E+m^2c^4\right) \\
p' &= \left(\frac{E'^2}{c^2} - m^2c^2\right)^{1/2} = \left(\frac{E^2}{4c^2} + \frac{mE}{2} + \frac{m^2c^2}{4} - m^2c^2\right)^{1/2} \\
&= \frac{E}{2c}\left(1+\frac{2mc^2}{E} - \frac{3m^2c^4}{E^2}\right)^{1/2} \approx \frac{E}{2c}\left(1+\frac{mc^2}{E} - \frac{3m^2c^4}{2E^2}\right) \approx \frac{E}{2c}\left(1+\frac{mc^2}{E}\right) \\
p_L &= \frac{1}{2}p = \frac{1}{2}\left(\frac{E^2}{c^2} - m^2c^2\right)^{1/2} = \frac{E}{2c}\left(1-\frac{m^2c^4}{E^2}\right)^{1/2} \approx \frac{E}{2c} \\
\cos\theta &= \frac{p_L}{p'} = \left(1+\frac{mc^2}{E}\right)^{-1} \approx 1 - \frac{mc^2}{E} \approx 1 - \frac{1}{2}\theta^2
\end{aligned}$$

Therefore, neglecting terms of order m^2/E^2, the (small) scattering angle is

$$\theta = \left(\frac{2mc^2}{E}\right)^{1/2}$$

PHYS3101 Analytical Mechanics Homework #11 Due 14 Nov 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) The MOLLER experiment at Jefferson Lab will measure elastic electron-electron, i.e. Møller, scattering with an electron beam impinging on a 125 cm long liquid hydrogen target. What is the target density of electrons in units of cm^{-2}? If the Møller scattering cross section is 60 μbarn, find the scattering rate if the beam current is 65 μA? (You will need to look up the density of liquid hydrogen.)

(2) The differential cross section for scattering 6.5-MeV α particles at 120° off a silver nucleus is about 0.5 barns/sr. If a total of 10^{10} α particles impinge on a silver foil of thickness 1 μm and if we detect the scattered particles using a counter of area 0.1 mm^2 at 120° and 1 cm from the target, about how many scattered α particles should we expect to count? Silver has a density of 10.5 g/cm^3, and atomic mass of 108.

(3) Our definition of the scattering cross section, $N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \sigma$ applies to an experiment using a narrow beam of projectiles all of which pass through a wide target assembly. Experimenters sometimes use a wide incident beam, which completely engulfs a small target assembly. Show that in this case $N_{\text{sc}} = n_{\text{inc}} N_{\text{tar}} \sigma$ where n_{inc} is the density (number/area) of the incident beam, viewed head-on, and N_{tar} is the total number of targets in the target assembly.

(4) A particle of mass m_1 and total energy $E = T + m_1 c^2$ scatters by an angle θ_{Lab} from a stationary particle of mass m_2. Use relativistic kinematics to derive an expression between the scattering angle θ_{CM} in the center-of-momentum, and θ_{lab}. In the case where the incident particle is non-relativistic ($T \ll m_1 c^2$) show that

$$\tan\theta_{\text{Lab}} = \frac{\sin\theta_{CM}}{\lambda + \cos\theta_{\text{CM}}} \qquad \text{where} \qquad \lambda \equiv \frac{m_1}{m_2}$$

This problem is more difficult than I thought. I can see my way through to a solution, but it is an arduous path and I don't think it is particularly enlightening. It's not so difficult, though, if you start with the assumption that the motion is non-relativistic, but this is essentially done in Taylor Section 14.8, which I didn't cover in class. So, let's skip this problem and this week you get a break, only four homework problems instead of five.

(5) Consider the non-relativistic scattering of two particles of equal mass. First, using the result of the previous problem, show that $\theta_{\text{Lab}} = \theta_{\text{CM}}/2$. Then prove that

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Lab}} = 4\cos\theta_{\text{Lab}} \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}}$$

Now, given that in the CM frame, the differential cross section is $R^2/4$ where $R = R_1 + R_2$ where R_1 and R_2 are the radii of the two masses, integrate over all directions to verify that the total cross section in the lab frame is πR^2, as it has to be.

(1) The density of liquid hydrogen is 70.85 g/L $= 7.1 \times 10^{-2}$ g/cm^3. There is one electron per hydrogen atom, so the target density is

$$n_{\rm tar} = \frac{\rho t}{m} = \frac{7.1 \times 10^{-2}\ {\rm g/cm^3} \times 125\ {\rm cm}}{1{\rm u} \times 1.66 \times 10^{-24}\ {\rm g/u}} = 5.3 \times 10^{24}/{\rm cm}^2$$

We want the rate, so using the notation we used in class, we write

$$\begin{aligned} R_{\rm sc} &= \frac{N_{\rm sc}}{T} = \frac{N_{\rm inc}}{T} n_{\rm tar}\sigma = \frac{I}{e} n_{\rm tar}\sigma \\ &= \frac{65 \times 10^{-6}\ {\rm C/s}}{1.6 \times 10^{-19}\ {\rm C}} \times 5.3 \times 10^{24}/{\rm cm}^2 \times 60 \times 10^{-6} \times 10^{-24}\ {\rm cm}^2 \\ &= 1.30 \times 10^{11}/{\rm s} = 130\ {\rm GHz} \end{aligned}$$

The MOLLER TDR gives 134 GHz, so this is good.

(2) This just uses are formula for scattered particles, namely

$$\begin{aligned} dN_{\rm sc} &= N_{\rm inc} n_{\rm tar}\sigma = N_{\rm inc}\frac{\rho t}{m} d\sigma = N_{\rm inc}\frac{\rho t}{m}\frac{d\sigma}{d\Omega} d\Omega = N_{\rm inc}\frac{\rho t}{m}\frac{d\sigma}{d\Omega}\frac{dA}{r^2} \\ &= 10^{10} \times \frac{10.5\ {\rm g/cm^3} \times 10^{-4}\ {\rm cm}}{108{\rm u} \times 1.66 \times 10^{-24}\ {\rm g/u}} \times 0.5 \times 10^{-24}\ {\rm cm}^2 \times \frac{10^{-3}\ {\rm cm}^2}{(1\ {\rm cm})^2} \\ &= 29 \end{aligned}$$

This agrees with the solution manual.

(3) This is pretty simple. To derive $N_{\rm sc} = N_{\rm inc} n_{\rm tar}\sigma$, we used a beam of area $A_{\rm beam}$ imbedded in a target having $N_{\rm tar}$ scattering centers spread over an area $A_{\rm tar} > A_{\rm beam}$. Therefore the number of scattering centers in the beam is $(N_{\rm tar}/A_{\rm tar})A_{\rm beam}$ and the total "scattering area" in the beam envelope is this times σ. The probability of scattering is the fraction of this scattering area over the beam envelope, so

$$N_{\rm sc} = N_{\rm inc}\frac{1}{A_{\rm beam}}\left[\frac{N_{\rm tar}}{A_{\rm tar}} A_{\rm beam}\sigma\right] = N_{\rm inc} n_{\rm tar}\sigma$$

where $n_{\rm tar} = N_{\rm tar}/A_{\rm tar}$ is the target density. Now if the situation is reversed and $A_{\rm tar} < A_{\rm beam}$, then the number of scattering centers is the total $N_{\rm tar}$ in the target. However, the fraction of the beam particles that can hit targets, in principle, is the fraction $N_{\rm inc}(A_{\rm tar}/A_{\rm beam})$. The total scattering area is just $N_{\rm tar}\sigma$ and the probability of scattering is this area divided by $A_{\rm tar}$, so the number of scattered particles is

$$N_{\rm sc} = N_{\rm inc}\frac{A_{\rm tar}}{A_{\rm beam}}\frac{N_{\rm tar}\sigma}{A_{\rm tar}} = n_{\rm inc} N_{\rm tar}\sigma$$

where $n_{\rm inc} = N_{\rm beam}/A_{\rm beam}$ is the beam density.

(4) I think you can do this problem by writing out the three different invariant dot products for the beam particle, the target particle, and the scattered beam particle. It looks like you get enough equations to solve for the relationship between the two angles, but it involves lots of messy square roots.

(5) For equal mass particles, the previous problem says

$$\tan\theta_{\rm Lab} = \frac{\sin\theta_{CM}}{1+\cos\theta_{\rm CM}} = \frac{2\sin\theta_{\rm CM}/2\cos\theta_{\rm CM}/2}{1+\cos^2\theta_{\rm CM}/2-\sin^2\theta_{\rm CM}/2} = \frac{2\sin\theta_{\rm CM}/2\cos\theta_{\rm CM}/2}{2\cos^2\theta_{\rm CM}/2} = \tan\frac{\theta_{\rm CM}}{2}$$

Therefore $\theta_{\rm Lab} = \theta_{\rm CM}/2$. Now the number of scattered events will be proportional to $d\sigma$, regardless of whether or not we are in the CM or lab frame, so

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Lab} d\Omega_{\rm Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} d\Omega_{\rm CM}$$

Since $d\Omega = \sin\theta\, d\theta\, d\phi$, and ϕ is the same in the CM and lab, we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM}\left|\frac{\sin\theta_{\rm CM}\,d\theta_{\rm CM}}{\sin\theta_{\rm Lab}\,d\theta_{\rm Lab}}\right| = \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM}\left|\frac{d\left(\cos\theta_{\rm CM}\right)}{d\left(\cos\theta_{\rm Lab}\right)}\right|$$

This is equation (14.45) in Taylor. Now it's easy to evaluate

$$\frac{d\left(\cos\theta_{\rm CM}\right)}{d\left(\cos\theta_{\rm Lab}\right)} = \frac{\sin\theta_{\rm CM}}{\sin\theta_{\rm Lab}}\frac{d\theta_{\rm CM}}{d\theta_{\rm Lab}} = \frac{\sin 2\theta_{\rm Lab}}{\sin\theta_{\rm Lab}}\frac{d(2\theta_{\rm Lab})}{d\theta_{\rm Lab}} = \frac{2\sin\theta_{\rm Lab}\cos\theta_{\rm Lab}}{\sin\theta_{\rm Lab}}2 = 4\cos\theta_{\rm Lab}$$

Therefore we arrive at

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Lab} = 4\cos\theta_{\rm Lab}\left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} = R^2\cos\theta_{\rm Lab}$$

Now for $0 \le \theta_{\rm CM} \le \pi$, we have $0 \le \theta_{\rm Lab} \le \pi/2$. The total cross section in the lab is therefore

$$\begin{aligned}\sigma_{\rm Lab} &= \int d\Omega\left(\frac{d\sigma}{d\Omega}\right)_{\rm Lab} = 2\pi\int_0^{\pi/2}\sin\theta_{\rm Lab}d\theta_{\rm Lab}\,R^2\cos\theta_{\rm Lab} \\ &= \pi R^2\int_0^{\pi/2}\sin 2\theta_{\rm Lab}\,d\theta_{\rm Lab} = \left[-\frac{1}{2}\pi R^2\cos 2\theta_{\rm Lab}\right]_0^{\pi/2} = \pi R^2\end{aligned}$$

PHYS3101 Analytical Mechanics Homework #12 Due 28 Nov 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A taut string has a fixed end at $x = -a$ and extends infinitely to positive x. The string is initially at rest and has the shape of an isosceles triangular pulse centered on $x = 0$ and extending over $-a/5 \leq x \leq a/5$. Find the equation for the shape of the string at all times, and plot it for $t = 0$, $t = 0.5a/c$, $t = a/c$, and $t = 1.5a/c$ where c is the speed of the wave on a string. If you prefer, you can create an animation. (Note that in MATHEMATICA, you can easily define this kind of function using HeavisideLambda.)

(2) In class we derived expressions for the kinetic and potential energies of a taut string as integrals over the length of the string. Using what you have previously learned for normal modes of the shape $u(x, t)$ of a stretched string of length L fixed to $u = 0$ at $x = 0$ and $x = L$, see Concepts (5.12), write the total energy of the string as a single sum over the normal modes and show that it is a constant in time.

(3) The wave equation in three spatial dimensions is

$$\boldsymbol{\nabla}^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

If $f(\mathbf{r}, t) = f(r, t)$ where r is the usual spherical polar coordinate, show that

$$f(r, t) = \frac{A}{r} e^{i(kr - \omega t)}$$

solves the wave equation, where A is a constant and $\omega = ck$.

(4) The equation of motion for an inviscid fluid of density $\rho(\mathbf{r}, t)$ in a gravitational field $\mathbf{g}$ is

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} p$$

where $\mathbf{v}(\mathbf{r}, t)$ is the velocity field and $p(\mathbf{r}, t)$ is the pressure field. Use this to show the familiar result from your first physics course that the difference in pressure between two points in a static and incompressible fluid separated by a vertical distance h is $\Delta p = \rho g h$.

(5) Find the speed of sound in air, using the following steps, and compare with the accepted value of 331 m/s at 0°C. First show that the bulk modulus of air is γp, where p is pressure and $\gamma \equiv C_p/C_V = 1.4$ is the ratio of specific heats for an ideal gas. You can assume that the adiabatic expansion and compression in air, as a sound wave passes, follows $pV^\gamma =$ constant, where V is the volume. Then use the ideal gas law $pV = NkT$ to express density of N air molecules with mass m in terms of p, m, and T. Finally, combine these two results using the formalism we developed in class. Repeat the calculation for helium gas instead of nitrogen, and explain why your voice sounds high pitched if you first breath in some helium before speaking.

PHYS3101 Analytical Mechanics Homework #12 Solutions

(1) See the accompanying MATHEMATICA notebook. If $h(x)$ be the initial shape, then

$$f(x,t) = \frac{1}{2}h(x-ct) + \frac{1}{2}h(x+ct)$$

satisfies the initial conditions $f(x,0) = h(x)$ and $\dot{f}(x,0) = 0$. As for the boundary condition at $x = -a$, the first term in $f(x,t)$ is a rightward moving so never encounters the left side of the string. We use a rightward moving virtual pulse to cancel the second term, so the full solution is

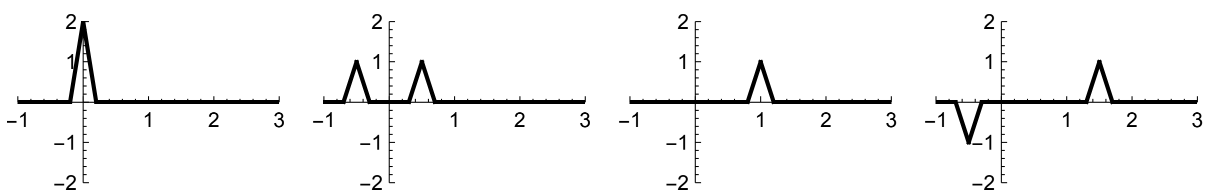
$$u(x,t) = f(x,t) - \frac{1}{2}h(x+2a-ct) = \frac{1}{2}h(x-ct) + \frac{1}{2}h(x+ct) - \frac{1}{2}h(x+2a-ct)$$

It is easy to see that the second and third terms cancel at $x = -a$. Define $y \equiv x + a$ so that $y = 0$ at the left end. Then

$$\frac{1}{2}h(x+ct) - \frac{1}{2}h(x+2a-ct) = \frac{1}{2}h(y-a+ct) - \frac{1}{2}h(y+a-ct) = \frac{1}{2}h(-a+ct) - \frac{1}{2}h(+a-ct)$$

at $y = 0$. But the function $h(x)$ is even, that is $h(-x) = h(x)$, so the final sum above is zero.

The plots below show the shape of the string at the four given times, plotted versus $\xi = x/a$:



The pulse splits in half, one part moving left and the other moving right. The leftward moving pulse reflects and reverses sign at the endpoint, and then both pieces move to the right forever. See the MATHEMATICA notebook for the animation.

(2) The expressions we derived in class for the kinetic and potential energies are

$$K = \frac{1}{2}\mu \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx \qquad \text{and} \qquad U = \frac{1}{2}T \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

where μ is the linear mass density of the string, T is the tension in the string, and $c = \sqrt{T/\mu}$ is the speed of waves on the string. We know that the normal mode solutions are a Fourier sine series, namely

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

We need to insert this into our expressions for K and U. Since the sums are squared, we need to write each of them using different indices, but we expect the integral over x to collapse the two sums to one because the sine and cosine functions are orthogonal. We have

$$\begin{aligned}
K &= \frac{1}{2}\mu \int_0^L \left[-\sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \right] \\
&\quad \times \left[-\sum_{m=1}^{\infty} B_m \left(\frac{m\pi c}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi ct}{L}\right) \right] dx \\
&= \frac{1}{2}\mu \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n B_m \left(\frac{n\pi c}{L}\right)\left(\frac{m\pi c}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{m\pi ct}{L}\right) \\
&\quad \times \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \\
&= \frac{1}{2}\mu \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n B_m \left(\frac{n\pi c}{L}\right)\left(\frac{m\pi c}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{m\pi ct}{L}\right) \frac{L}{2}\delta_{nm} \\
&= \frac{1}{4}\mu L \sum_{n=1}^{\infty} B_n^2 \left(\frac{n\pi c}{L}\right)^2 \sin^2\left(\frac{n\pi ct}{L}\right)
\end{aligned}$$

and

$$\begin{aligned}
U &= \frac{1}{2}T \int_0^L \left[\sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \right] \\
&\quad \times \left[\sum_{m=1}^{\infty} B_m \left(\frac{m\pi}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{m\pi ct}{L}\right) \right] dx \\
&= \frac{1}{2}T \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n B_m \left(\frac{n\pi}{L}\right)\left(\frac{m\pi}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \cos\left(\frac{m\pi ct}{L}\right) \\
&\quad \times \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx \\
&= \frac{1}{2}T \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_n B_m \left(\frac{n\pi}{L}\right)\left(\frac{m\pi}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \cos\left(\frac{m\pi ct}{L}\right) \frac{L}{2}\delta_{nm} \\
&= \frac{1}{4}TL \sum_{n=1}^{\infty} B_n^2 \left(\frac{n\pi}{L}\right)^2 \cos^2\left(\frac{n\pi ct}{L}\right)
\end{aligned}$$

Therefore, writing $T = c^2\mu$, the total energy is

$$E = T + U = \frac{1}{4}\mu c^2 L \sum_{n=1}^{\infty} B_n^2 \left(\frac{n\pi}{L}\right)^2 \left[\sin^2\left(\frac{n\pi ct}{L}\right) + \cos^2\left(\frac{n\pi ct}{L}\right)\right] = \frac{1}{4}\mu c^2 L \sum_{n=1}^{\infty} B_n^2 \left(\frac{n\pi}{L}\right)^2$$

which is indeed constant in time. It is worth checking that this result is dimensionally correct. The B_n have units of length, so the quantity in the sum is dimensionless, but μL has dimensions of mass, so the factor out front has dimensions of mass times velocity^2 which is energy.

(3) It is easiest to work this out using the Laplacian in spherical coordinates. That is, using (4.26c) in Concepts, but start with a rearrangement of the radial derivatives, namely

$$\boldsymbol{\nabla}^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r}\left(2\frac{\partial f}{\partial r} + r\frac{\partial^2 f}{\partial r^2}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) = \frac{A}{r}(-k^2)e^{i(kr-\omega t)}$$

We also need to calculate

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{c^2}(-\omega^2)\frac{A}{r}e^{i(kr-\omega t)}P_\ell(\cos\theta) = -k^2\frac{A}{r}e^{i(kr-\omega t)}P_\ell(\cos\theta)$$

These two expressions are equal, so subtracting them (to give the wave equation) gives zero.

(4) This is simple. The fluid is static, so $d\mathbf{v}/dt = 0$ and ρ is a constant for an incompressible fluid. Writing $\mathbf{g} = -g\hat{\mathbf{z}}$, we see explicitly that p only changes in the z-direction, so we have

$$\boldsymbol{\nabla}p = \hat{\mathbf{z}}\frac{dp}{dz} = -\rho g\hat{\mathbf{z}} \qquad \text{therefore} \qquad \Delta p = -\rho g\Delta z$$

where the $-$ sign only means that the pressure increases as the depth increases. The (absolute value) of the difference in pressure between two points separated by a height $\Delta z = h$ is therefore $\rho g h$.

(5) The bulk modulus E is defined by $dp = -E(dV/V)$, but $d(pV^\gamma) = V^\gamma dp + \gamma p V^{\gamma-1}dV = 0$ since pV^γ is constant. So $dp = -\gamma p dV/V$ and $E = \gamma p$. Now $\rho = Nm/V = mp/kT$ so the speed of sound c is given by $c^2 = E/\rho = \gamma kT/m$. (I use "physics" quantities instead of "chemistry.") Now $\gamma = 1.4$, $k = 1.38 \times 10^{-23}$J/K, and $T = 273$K. Air is mostly diatomic nitrogen molecules, with $\approx 20\%$ oxygen, so take $m = (0.8\cdot 28 + 0.2\cdot 32)m_p = 28.8m_p$ where m_p is the proton mass. With $m_p = 1.67 \times 10^{-27}$ kg, have $m = 4.81 \times 10^{-26}$ kg and $c = 331$ m/s. In helium gas the speed is faster by a factor $(28.8/4)^{1/2} \approx 2.7$. This vibrates your vocal chords at a higher rate, so your voice sounds high pitched.

PHYS3101 Analytical Mechanics   Homework #13   Due 5 Dec 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

This assignment is on nonlinear dynamics and chaos as demonstrated by the Damped Driven Pendulum (DDP). You should execute, and play around, with the notebook provided on the course web page that goes with Chapter 12 in Taylor. You are welcome to borrow code from that notebook for this assignment.

(1) Reproduce Figure 12.7 in Taylor, namely two solutions for the DDP with the same drive strength and damping parameter, and the initial condition $\dot{\phi}(0) = 0$, but one solution for $\phi(0) = 0$ and the other for $\phi(0) = -\pi/2$. This example demonstrates that for a nonlinear system, the behavior can be wildly different for different initial conditions. Animate the two solutions, and compare them. (It would be more fun to do this with another person, and start the two animations at the same time to watch and compare in real time.)

(2) Using the code that reproduces Figure 12.4 in Taylor, a DDP with our standard frequencies and damping parameter and with drive strength $\gamma = 1.06$, find solutions for the two initial conditions $\phi(0) = \pi/2$ and $\phi(0) = -\pi/2$, both with $\dot{\phi}(0) = 0$. Plot all three solutions for $0 \le t \le 10$, or longer. Do all three approach the same solution after some period of time? You may need to remember that $\phi(t)$ is the same as $\phi(t) + 2\pi n$ for some integer n.

(3) The notebook for class demonstrates chaotic behavior when the drive strength $\gamma = 1.105$ with our other standard parameters. Increasing the drive strength to $\gamma = 1.503$ (Taylor Figures 12.15(a), 12.15(b), and 12.16) continues chaotic motion, but the motion is qualitatively very different. Reproduce these three figures. (Note that Figure 12.15 uses $\phi(0) = -\pi/2$.) You might see deviation from Taylor's figure after long times because of numerical precision, but you can consider using the option PrecisionGoal in NDSolve. Make an animation of these conditions, and watch the pendulum flip directions near $t \approx 17$.

(4) It happens that periodicity can be restored with driving strengths well past the onset of chaos. Set up and solve the DPP using a driving strength $\gamma = 1.3$ and our standard frequencies and damping parameter, with initial conditions $\phi(0) = \dot{\phi}(0) = 0$. Plot the solution and comment. (The animation might be fun to watch.) Show that the solution is in fact periodic at long times. You may need to subtract a linear function that looks something like $2\pi(t - t_0)$ to make the periodicity clear.

(5) This problem is an example of a *logistics map*, a mathematical example of nonlinearity which shows many of the same features as the DDP. See Taylor 12.9. Consider a set of numbers $\{x_0, x_1, x_2, \ldots, x_\infty\}$ is defined by the "sine map" $x_{i+1} = f(x_i)$ where $f(x) = r\sin(\pi x)$, an obviously nonlinear function. Find and plot the values of x_i for i up to some number, say $i_{\max} = 20$ to start, for $x_0 = 0.8$ and $r = 0.60$, 0.79, 0.85, and 0.865, and show that these values of r form a period-doubling cascade, similar to what happens in Figure 12.8 in Taylor. (You will likely find it useful to use the RecurrenceTable function in Mathematica.)

PHYS3101 Analytical Mechanics Homework #13 Solutions

All of the solutions for this assignment are in the associated Mathematica notebook.