PHYS3101 Analytical Mechanics Fall 2023

Final Exam Thursday 14 Dec 2023

There are **five questions** and you are to work all of them. You may use the textbook, your own notes, or any other resources, but you may not communicate with another human. Of course, if you have questions, you are encouraged to ask the person proctoring the exam.

The five problems will be equally weighted. If you are stuck on one, move on to another and come back if you have time.

Please start each problem on a new page in your exam booklet.

Good luck!

(1) A long, thin, uniform rigid rod of mass m and length L hangs from the ceiling by one end, and swings freely through a vertical plane. If θ measures the angle with respect to the vertical, where $\theta = 0$ means that the rod hangs straight down, find the Lagrangian $\mathcal{L}(\theta, \dot{\theta})$ in terms of m, L, and g. Then use this to find the differential equation of motion for θ. Finally, find the frequency of oscillations for $\theta \ll 1$.

(2) A particular stress tensor at some point (x, y, z) in some medium is given by

$$\underline{\underline{\Sigma}} = \frac{\sigma}{L^2} \begin{bmatrix} xz & z^2 & 0 \\ z^2 & 0 & -xy \\ 0 & -xy & 0 \end{bmatrix}$$

where σ and L are constants. Find the force on a small area element dA that lies in the plane $x + 2y + 2z = 0$ in the region of the point $(x, y, z) = (4L, -L, -L)$.

(3) A particle of mass m moves without friction on a horizontal surface, and is attached to the origin by a spring with stiffness k. Find the Hamiltonian $\mathcal{H}(r, p_r, \phi, p_\phi)$ using plane polar coordinates (r, ϕ). Show that the Hamiltonian reduces to a function $\mathcal{H}(r, p_r)$ in terms of m, k, and a single additional constant. Use this result to find the radius R of a stable circular orbit, and show that the result agrees with what you would have expected from the "$F = ma$" approach in polar coordinates.

(4) Two identical pendulums made from massless strings with length L and bob mass m swing in the same plane and hang from a horizontal ceiling. The bobs are attached to each other by a spring with stiffness k. The equilibrium length of the spring is the same as the distance between the two points at which the pendulums are connected to the ceilings. Assuming only small amplitudes, find the normal mode frequencies in terms of g, L, k, and m, and describe the motions of the normal modes. (You are welcome to use your choice of Newtonian, Lagrangian, or Hamiltonian dynamics to set up the equations of motion.)

(5) A (massless) photon of energy E collides with a stationary particle of mass m. The particle of mass m remains after the collision, but the photon disappears and a new particle of mass M appears. Find the minimum value of E for which this reaction is possible, in terms of m, M, and the speed of light c.

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Solutions

(1) The kinetic energy is $T = I\dot{\theta}^2/2$ where $I = mL^2/3$ and the potential energy is $U = mgh$ where $h = L(1 - \cos\theta)/2$ is the height of the center of mass. Therefore

$$\mathcal{L}(\theta, \dot{\theta}) = T - U = \frac{1}{6}mL^2\dot{\theta}^2 - \frac{1}{2}mgL(1 - \cos\theta)$$

The equation of motion is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} \qquad \text{so} \qquad \frac{1}{3}mL^2\ddot{\theta} = -\frac{1}{2}mgL\sin\theta \qquad \text{or} \qquad \ddot{\theta} = -\frac{3}{2}\frac{g}{L}\sin\theta$$

The frequency of small oscillations is then $\omega = \sqrt{3g/2L}$.

(2) Written in terms of matrices and column vectors, the force $d\vec{F}$ on an area element $d\vec{A}$ is given by $d\underline{F} = \underline{\underline{\Sigma}}\, d\underline{A}$ where we evaluate the elements of $\underline{\underline{\Sigma}}$ at the given point, and the elements of $d\underline{A}$ are determined from the normal vector $\hat{n}$ of the plane. From the equation of the plane, we know that $\hat{n}$ is proportional to the $(1, 2, 2)$ direction. Normalizing this gives

$$\hat{n} = \frac{1}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k} \qquad \text{so} \qquad d\underline{F} = \frac{\sigma}{L^2}\begin{bmatrix} -4L^2 & L^2 & 0 \\ L^2 & 0 & 4L^2 \\ 0 & 4L^2 & 0 \end{bmatrix} dA \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \frac{\sigma dA}{3}\begin{bmatrix} -2 \\ 9 \\ 8 \end{bmatrix}$$

In other words, written in the usual vector notation in three spacial dimensions,

$$d\vec{F} = \frac{\sigma dA}{3}\left(-2\hat{\imath} + 9\hat{\jmath} + 8\hat{k}\right)$$

(3) Follow the standard procedure to find the Hamiltonian:

$$\begin{aligned}
\mathcal{L}(r,\dot{r},\phi,\dot{\phi}) &= T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{1}{2}kr^2 \\
p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \qquad \text{so} \qquad \dot{r} = \frac{p_r}{m} \\
p_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} \qquad \text{so} \qquad \dot{\phi} = \frac{p_\phi}{mr^2} \\
\mathcal{H}(r,p_r,\phi,p_\phi) &= \dot{r}p_r + \dot{\phi}p_\phi - \mathcal{L} \\
&= \frac{p_r^2}{m} + \frac{p_\phi^2}{mr^2} - \frac{1}{2}m\frac{p_r^2}{m^2} - \frac{1}{2}mr^2\frac{p_\phi^2}{m^2r^4} + \frac{1}{2}kr^2 \\
&= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{2}kr^2
\end{aligned}$$

Now we know that

$$\dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0 \qquad \text{so} \qquad p_\phi = \text{constant} \equiv \ell$$

which is just the angular momentum of the particle. Therefore

$$\mathcal{H}(r,p_r,\phi,p_\phi) = \frac{p_r^2}{2m} + \frac{\ell^2}{2mr^2} + \frac{1}{2}kr^2 = \mathcal{H}(r,p_r)$$

Since the Langrangian does not depend explicitly on time, the Hamiltonian is a conserved quantity, in fact equal to the total energy. So we write

$$E = \frac{p_r^2}{2m} + U_{\text{eff}}(r) \qquad \text{where} \qquad U_{\text{eff}}(r) = \frac{\ell^2}{2mr^2} + \frac{1}{2}kr^2$$

is an effective radial potential energy function. It is clear that $U_{\text{eff}}(r)$ has a local minimum at some $r = R$ because it tends to infinity for $r \to \infty$ and $r \to 0$. Therefore we have

$$U'_{\text{eff}}(R) = -\frac{\ell^2}{mR^3} + kR = 0 \qquad \text{so} \qquad R = \left(\frac{\ell^2}{mk}\right)^{1/4}$$

If we were doing this problem in a Physics I class, we would have written something like

$$F = -kR = ma = m\left(-\frac{v^2}{R}\right) = -\frac{m^2v^2R^2}{mR^3} \qquad \text{that is} \qquad kR = \frac{\ell^2}{mR^3}$$

which gives the same result for R as when we used Hamiltonian dynamics.

(4) This problem is very similar to what is covered in Taylor Section 11.6, so we'll follow that approach and use Lagrangian dynamics. Labeling the pendulums as 1 and 2, with angles ϕ, the kinetic energy of each is just $mL^2\dot{\phi}^2/2$. Each pendulum also has a gravitational potential energy $mgL(1-\cos\phi) \approx mgL\phi^2/2$. There is also the potential energy stored in the spring, that is $k(x_2-x_1)^2/2 = k(L\sin\phi_1 - L\sin\phi_2)^2/2 \approx kL^2(\phi_2-\phi_1)^2/2$. Therefore

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\phi}_1^2 + \frac{1}{2}mL^2\dot{\phi}_2^2 - \frac{1}{2}mgL\phi_1^2 - \frac{1}{2}mgL\phi_2^2 - \frac{1}{2}kL^2(\phi_2-\phi_1)^2$$

This Lagrangian will give us an equation of motion

$$\underline{\underline{M}}\,\underline{\ddot{\phi}} + \underline{\underline{K}}\,\underline{\phi} = 0 \quad \text{where} \quad \underline{\underline{M}} = mL^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \underline{\underline{K}} = \begin{bmatrix} mgL+kL^2 & -kL^2 \\ -kL^2 & mgL+kL^2 \end{bmatrix}$$

Inserting our usual ansatz leads us to $|\underline{\underline{K}} - \omega^2\underline{\underline{M}}| = 0$ which reduces to

$$\begin{vmatrix} g/L + k/m - \omega^2 & -k/m \\ -k/m & g/L + k/m - \omega^2 \end{vmatrix} = \left(\frac{g}{L} + \frac{k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

The eigenfrequencies are therefore

$$\omega^2 = \frac{g}{L} + \frac{k}{m} \mp \frac{k}{m} = \frac{g}{L} \qquad \text{or} \qquad = \frac{g}{L} + 2\frac{k}{m}$$

It is easy to see that inserting the first solution into $\underline{\underline{K}} - \omega^2\underline{\underline{M}}$ gives equal (in magnitude and sign) amplitudes for ϕ_1 and ϕ_2. This makes perfect sense, since the frequency is just that of a simple pendulum because the spring never stretches, and the two pendulums just swing together in unison at the frequency of a single simple pendulum, namely $\sqrt{g/L}$.

It is also easy to see that the second solution gives the two pendulums swinging against each other, with equal amplitudes but opposite signs. In this case, the frequency is larger with an effective spring constant $2k$ because there is a "double" force from the spring because it is stretched and compressed from two directions.

(5) Write k, p_0, p, and P for the four-momenta of the photon, target, and final state particles m and M, respectively. Conservation of four-momentum allows us to write

$$(k + p_0)^2 = (p + P)^2$$

where we'll evaluate the left in the lab frame, which is simple and which gives us an expression in terms of E, and the right in the CM frame, which makes it easy to write down the value at threshold. For the left hand side, we have

$$k = \left(\frac{E}{c}, \frac{\vec{E}}{c}\right) \qquad \text{and} \qquad p_0 = (mc, \vec{0})$$

Therefore

$$(k + p_0)^2 = 2k \cdot p_0 + m^2c^2 = 2Em + m^2c^2$$

For the right hand side, at threshold, the three momenta are zero, so we just have

$$(p + P)^2 = (mc + Mc, \vec{0})^2 = (m + M)^2c^2 = m^2c^2 + 2mMc^2 + M^2c^2$$

Equating both sides gives

$$E = \frac{1}{2m}(2mMc^2 + M^2c^2) = Mc^2 + \frac{M^2c^4}{2mc^2} = Mc^2\left[1 + \frac{M}{2m}\right]$$

It's not asked in the problem, but we can check some things. The limit $m \gg M$ (including $M = 0$) gives the expected answer. We can also put in some numbers and check against experiment, for example the reaction $\gamma p \to J/\psi + p$, for which the threshold is given as $E = 8.2$ GeV. (See S. Adhikari et al., Phys. Rev. C 108(2023)025201.) We get

$$E = 3.1 \text{ GeV} + \frac{(3.1 \text{ GeV})^2}{2 \times 0.93 \text{ GeV}} = 8.2 \text{ GeV}$$

Note that if you want to build a device that produces photons that can make M, and $M \gg m$, then the threshold photon energy rises as the *square* of M. This presents a challenge to so-called "fixed target" experiments that want to produce massive particles.