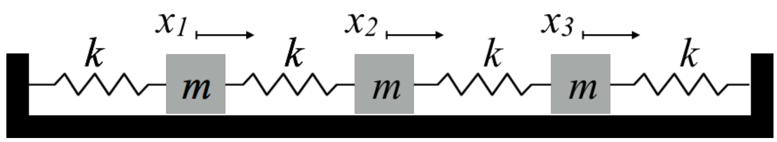
Physics with Mathematica Fall 2019 Exercise #6 2 Oct 2019

Three masses and four springs as an eigenvalue problem

The real point of this problem is to help you get a feel for what eigenvectors and eigenvalues mean for a familiar physical system.

Consider the system of a one dimensional oscillator assembled from three identical masses connected by four identical springs, as shown here:



First, solve this problem as a system of differential equations for $x_1(t)$, $x_2(t)$, and $x_3(t)$, that is, $m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$, and so on. Use initial conditions where all three masses are at rest, x_1 is displaced an amount a, and both x_2 and x_3 are at their equilibrium positions.

With the substitution $k = m\omega_0^2$, plot the motions $x_1(t)$, $x_2(t)$, and $x_3(t)$ for $\omega_0 = 2\pi$ (i.e. $2\pi/\omega_0 = 1$) and $a = 1$. Does the motion seem periodic? You may need to plot up to a relatively large value of t in order to see the motions repeat.

Display the expressions you have for $x_1(t)$, $x_2(t)$, and $x_3(t)$, and you should be able to easily identify the three different frequencies involved. The functions ComplexExpand and Simplify may be helpful. Don't forget to use $Assumptions to specify that $\omega_0 > 0$.

Now treat the system as an eigenvalue problem. That is, write the equations of motion as $\mathsf{M}\ddot{\mathsf{x}} = -\mathsf{K}\mathsf{x}$ where M and K are 3×3 matrices, and x is a column vector. Then write $\mathsf{x} = \mathsf{a}e^{i\omega t}$ so that $\Omega\mathsf{a} = \omega^2\mathsf{a}$ where Omega is a 3×3 matrix in terms of ω_0^2 and a is a column vector. In other words, you now want to find the eigenvalues ω^2 of Ω, along with the eigenvectors a. The eigenvalues should agree with the expressions you found for $x_1(t)$, $x_2(t)$, and $x_3(t)$.

Note: The matrix M *is trivial. You are welcome to enter the elements of* K *by hand, but if you are ambitious, try obtaining it by inserting* $\mathsf{x} = \mathsf{a}e^{i\omega t}$ *into the equations, then using* Part *to extract right side of the equations, and finally using* Coefficient *to get the elements of* K.

Finally, form the linear combinations $x_a(t)$, $x_b(t)$, and $x_c(t)$ of $x_1(t)$, $x_2(t)$, and $x_3(t)$, where the coefficients are the three sets of eigenvectors you found for a. Plot $x_a(t)$, $x_b(t)$, and $x_c(t)$ and show that they are obviously periodic, and display the expressions to show that they have periods given by the eigenvalues.

Send the grader an email with your notebook as an attachment.