Name: ____________________

PHYS2502 Mathematical Physics   Quiz #1   13 Jan 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

The so-called "Planck Mass" is the mass (or energy, when multiplied by c^2) at which gravity is unified with quantum mechanics and special relativity. Therefore, it must depend on (the reduced) Planck's constant $\hbar$, the speed of light c, and Newton's gravitational constant G. Use dimensional analysis to arrive at an expression for the Planck Mass M_P. It will help to know that $\hbar$ has units of angular momentum, and that the force of gravity between two masses m_1 and m_2 separated by a distance r is Gm_1m_2/r^2.

PHY2502 Mathematical Physics Quiz #1 Solution

The so-called "Planck Mass" is the mass (or energy, when multiplied by c^2) at which gravity is unified with quantum mechanics and special relativity. Therefore, it must depend on (the reduced) Planck's constant $\hbar$, the speed of light c, and Newton's gravitational constant G. Use dimensional analysis to arrive at an expression for the Planck Mass M_P. It will help to know that $\hbar$ has units of angular momentum, and that the force of gravity between two masses m_1 and m_2 separated by a distance r is Gm_1m_2/r^2.

$$[\hbar] = L \cdot MLT^{-1} = L^2MT^{-1}$$

$$[\text{force}] = MLT^{-2} = [G]M^2L^{-2} \qquad \text{so} \qquad [G] = L^3M^{-1}T^{-2}$$

Writing $M_P = G^x\hbar^y c^z$ means that

$$M = L^{3x}M^{-x}T^{-2x}L^{2y}M^yT^{-y}L^zT^{-z} = L^{3x+2y+z}M^{-x+y}T^{-2x-y-z}$$

The mass factor says that $y = x + 1$, so the length and time factors together say that

$$\begin{aligned} 3x + 2y + z = 5x + 2 + z = 0 \qquad \text{so} \qquad 5x + z &= -2 \\ -2x - y - z = -3x - 1 - z = 0 \qquad \text{so} \qquad 3x + z &= -1 \end{aligned}$$

Subtract these to get $2x = -1$ or $x = -1/2$. Then $y = x + 1 = 1/2$. There are several equations that can give you z, for example $z = --1 - 3x = -1 + 3/2 = 1/2$. Therefore

$$M_P = G^{-1/2}\hbar^{1/2}c^{1/2} = \left(\frac{\hbar c}{G}\right)^{1/2}$$

which is verified easily enough with a quick Google search.

Name: ________________________________

PHYS2502 Mathematical Physics Quiz #2 27 Jan 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

This quiz has two parts, each worth 10 points.

(1) Suppose you have 100 m of fencing and you want to create a rectangular pen for your dog. You have a very long wall that you can use for one side of the pen, and you want to make a pen with the largest area as possible for your dog to run around in. What should you use for the length and width of the pen?

(2) For the function $y = \cos^{-1}(x)$, find the derivative dy/dx. I think the simplest way to do this is to make use of differentials.

PHY2502 Mathematical Physics Quiz #2 Solution

This quiz has two parts, each worth 10 points.

(1) Suppose you have 100 m of fencing and you want to create a rectangular pen for your dog. You have a very long wall that you can use for one side of the pen, and you want to make a pen with the largest area as possible for your dog to run around in. What should you use for the length and width of the pen?

(2) For the function $y = \cos^{-1}(x)$, find the derivative dy/dx. I think the simplest way to do this is to make use of differentials.

(1) Let x be the width of the pen, and y be the length, with the long side parallel to the wall. You have a constraint that $2x + y = 100$. The area of the pen is $A = xy$. So, to maximize A, calculate

$$\frac{dA}{dx} = \frac{d}{dx}x(100 - 2x) = \frac{d}{dx}(100x - 2x^2) = 100 - 4x = 0$$

Therefore $x = 25$ m, implying $y = 50$ m, maximizes the area of the pen.

(2) We don't know the derivative of the $\cos^{-1}$ function, but we do know the derivative of cos, so write $\cos(y) = x$ and therefore $-\sin(y)\,dy = dx$ which gives

$$\frac{dy}{dx} = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1 - \cos^2(y)}} = -\frac{1}{\sqrt{1 - x^2}}$$

Name: ______________________________

PHYS2502 Mathematical Physics Quiz #3 3 Feb 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

The energy of a relativistic particle of (rest) mass m is

$$E = \sqrt{p^2c^2 + m^2c^4}$$

where p is the momentum and c is the speed of light. Find an expression for E when $p \ll mc$ to lowest nonzero order in p and physically interpret the terms in the expression.

PHY2502 Mathematical Physics Quiz #3 Solution

The energy of a relativistic particle of (rest) mass m is

$$E = \sqrt{p^2c^2 + m^2c^4}$$

where p is the momentum and c is the speed of light. Find an expression for E when $p \ll mc$ to lowest nonzero order in p and physically interpret the terms in the expression.

$$E = \left(p^2c^2 + m^2c^4\right)^{1/2} = mc^2\left(1 + \frac{p^2}{m^2c^2}\right)^{1/2} = mc^2\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2} + \cdots\right) = mc^2 + \frac{p^2}{2m}$$

The first term is the (famous) "rest energy" mc^2 of a particle with mass m. The second term is just the (nonrelativistic) kinetic energy for a particle of momentum $p = mv$ and mass m, that is $p^2/2m = mv^2/2$.

Name: ______________________________

PHYS2502 Mathematical Physics   Quiz #4   10 Feb 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the solution $y(x)$ to the first order linear differential equation

$$y'(x) = x + y(x) \qquad \text{with} \qquad y(0) = 0$$

PHY2502 Mathematical Physics Quiz #4 Solution

Find the solution $y(x)$ to the first order linear differential equation

$$y'(x) = x + y(x) \qquad \text{with} \qquad y(0) = 0$$

Writing the equation as $y'(x) - y(x) = x$ makes it clear that this is a simple example of the form $y'(x) + p(x)y(x) = q(x)$ where $p(x) = -1$ and $q(x) = x$. The integrating factor $\exp[\int p(x)dx] = e^{-x}$, so we aim to integrate both sides of

$$e^{-x}y'(x) - e^{-x}y(x) = xe^{-x}$$

Indeed, the left side is just the derivative of $e^{-x}y(x)$. Using integration by parts on the right

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Therefore, integrating the whole equation, we get

$$e^{-x}y(x) = -xe^{-x} - e^{-x} + C \qquad \text{or} \qquad y(x) = Ce^{x} - x - 1$$

Since $y(0) = C - 1 = 0$ we have $C = 1$. Therefore the complete solution is

$$y(x) = e^{x} - x - 1$$

Name: ____________________

PHYS2502 Mathematical Physics   Quiz #5   17 Feb 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the solution $y(x)$ to the second order linear differential equation

$$y''(x) + y(x) = \sin(2x) \qquad \text{with} \qquad y(0) = 0,\, y'(0) = 0$$

PHY2502 Mathematical Physics Quiz #5 Solution

Find the solution $y(x)$ to the second order linear differential equation

$$y''(x) + y(x) = \sin(2x) \qquad \text{with} \qquad y(0) = 0,\, y'(0) = 0$$

The solution to the homogeneous equation is

$$y_H(x) = a\cos x + b\sin x$$

where a and b are arbitrary constants. It's pretty easy to guess the form of a particular solution, namely $y_P(x) = c\sin(2x)$. Inserting into the differential equation we have

$$-4c\sin(2x) + c\sin(2x) = \sin(2x) \qquad \text{so} \qquad c = -\frac{1}{3}$$

Therefore the general solution is

$$y(x) = y_H(x) + y_P(x) = a\cos x + b\sin x - \frac{1}{3}\sin(2x)$$

Applying the initial conditions gives us

$$\begin{aligned} a(1) + b(0) - \frac{1}{3}(0) &= 0 \\ -a(0) + b(1) - \frac{2}{3}(1) &= 0 \end{aligned}$$

which means that $a = 0$ and $b = 2/3$. Therefore the solution is

$$y(x) = \frac{2}{3}\sin x - \frac{1}{3}\sin(2x)$$

Writing $\sin(2x) = 2\sin x\cos x$ gives us the alternate form

$$y(x) = \frac{2}{3}\sin x\,(1 - \cos x)$$

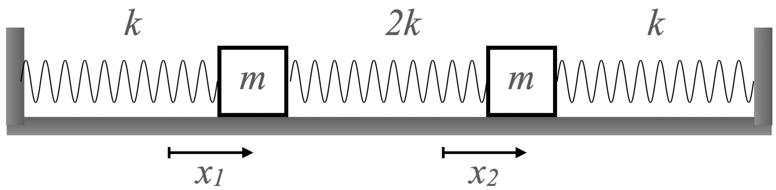
Name: ______________________

PHYS2502 Mathematical Physics Quiz #6 24 Feb 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

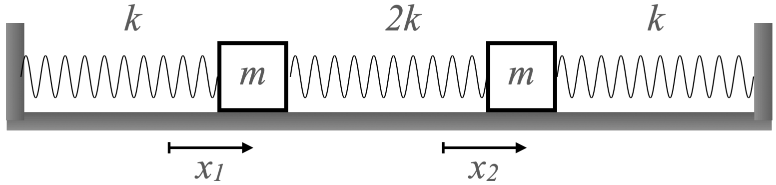
Two masses and three springs are arranged as shown here:



Find the eigenfrequencies for the two eigenmodes of oscillation, and determine the relative amplitudes of oscillation of the two masses for each mode.

PHY2502 Mathematical Physics Quiz #6 <u>Solution</u>

Two masses and three springs are arranged as shown here:



Find the eigenfrequencies for the two eigenmodes of oscillation, and determine the relative amplitudes of oscillation of the two masses for each mode.

This is very similar to the problem we worked in class, just with a different spring constant for the middle (coupling) spring. We expect to have one mode where the coupling spring stays the same length at all times, so the oscillation frequency is $\omega_0^2 = k/m$. The second mode should be the one where the two masses oscillate against each other, 180° out of phase, but with a higher frequency than $\omega^2 = 3\omega_0^2$ that we had previously, because the coupling spring is stiffer. But let's work it out and see.

Newton's Second Law for each of the two masses becomes

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + 2k(x_2 - x_1) \\ m\ddot{x}_2 &= -kx_2 - 2k(x_2 - x_1) \end{aligned}$$

Insert the standard anszatz $x_1 = a_1 e^{i\omega t}$ and $x_1 = a_1 e^{i\omega t}$ to get

$$(3\omega_0^2 - \omega^2)a_1 - 2\omega_0^2 a_2 = 0 \qquad \text{and} \qquad -2\omega_0^2 a_1 + (3\omega_0^2 - \omega^2)a_2 = 0$$

Forcing the ratio of the coefficients to be the same gives us

$$(3\omega_0^2 - \omega^2)^2 = 4\omega_0^4 \qquad \text{so} \qquad 3\omega_0^2 - \omega^2 = \pm 2\omega_0^2$$

Therefore, the (squares of the) eigenfrequencies are $\omega^2 = \omega_0^2$ and $\omega^2 = 5\omega_0^2$, perfectly consistent with what we predicted. For the mode with $\omega = \omega_0$, we find

$$2\omega_0^2 a_1 - 2\omega_0^2 a_2 = 0 \qquad \text{so} \qquad a_2 = a_1$$

and indeed the two masses oscillate together with the same amplitude and in phase. On the other hand, for the mode with $\omega = \sqrt{5}\omega_0$, we get

$$-2\omega_0^2 a_1 - 2\omega_0^2 a_2 = 0 \qquad \text{so} \qquad a_2 = -a_1$$

and again, as predicted, the two masses oscillate together with the same amplitude but 180° out of phase.

Name: ______________________________

PHYS2502 Mathematical Physics Quiz #7 8 Mar 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

(1) Calculate the gradient of the scalar field $f(x, y, z) = xyz$.

(2) Calculate the divergence of the vector field $\vec{v}(x, y, z) = \hat{\imath}x + \hat{\jmath}z - \hat{k}y$.

(3) Calculate the curl of the vector field $\vec{v}(x, y, z) = \hat{\imath}x + \hat{\jmath}z - \hat{k}y$.

PHY2502 Mathematical Physics Quiz #7 Solution

(1) Calculate the gradient of the scalar field $f(x, y, z) = xyz$.

(2) Calculate the divergence of the vector field $\vec{v}(x, y, z) = \hat{\imath}x + \hat{\jmath}z - \hat{k}y$.

(3) Calculate the curl of the vector field $\vec{v}(x, y, z) = \hat{\imath}x + \hat{\jmath}z - \hat{k}y$.

(1) Gradient.

$$\vec{\nabla} f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = \hat{\imath}yz + \hat{\jmath}xz + \hat{k}xy$$

(2) Divergence.

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 1 + 0 + 0 = 1$$

(3) Curl.

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= \hat{\imath}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\ &= \hat{\imath}(-1 - 1) + \hat{\jmath}(0 - 0) + \hat{k}(0 - 0) = -2\hat{\imath} \end{aligned}$$

Name: ____________________

PHYS2502 Mathematical Physics Quiz #8 24 Mar 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

This quiz has two parts, each worth 10 points.

(1) Find the solution $u(x, y)$ for the partial differential equation and boundary condition

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} \qquad u(x, 0) = 2x$$

(2) Find the Fourier Transform $A(k)$ of the function $f(x) = ae^{-\beta|x|}$ were $\beta > 0$. You are welcome to make use of the definite integral

$$\int_0^\infty e^{-\beta x} \cos(kx)\, dx = \frac{\beta}{\beta^2 + k^2}$$

PHY2502 Mathematical Physics Quiz #8 Solution

This quiz has two parts, each worth 10 points.

(1) Find the solution $u(x,y)$ for the partial differential equation and boundary condition

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y} \qquad u(x,0) = 2x$$

(2) Find the Fourier Transform $A(k)$ of the function $f(x) = ae^{-\beta|x|}$ were $\beta > 0$. You are welcome to make use of the definite integral

$$\int_0^\infty e^{-\beta x}\cos(kx)\,dx = \frac{\beta}{\beta^2+k^2}$$

(1) By inspection, the general solution is $u(x,y) = f(x-y)$ for some function $f(z)$. The boundary condition gives $u(x,0) = f(x) = 2x$. Therefore, the solution is $u(x,y) = 2(x-y)$.

(2) Just apply the definition of the Fourier Transform and make use of the fact that $f(x)$ is an even function, as is $\cos(kx)$, whereas $\sin(kx)$ is odd. You have

$$\begin{aligned} A(k) &= \int_{-\infty}^{\infty} e^{-ikx} f(x) = \int_{-\infty}^{\infty} [\cos(kx) - i\sin(kx)]\, ae^{-\beta|x|}\,dx \\ &= 2a\int_0^\infty \cos(kx)e^{-\beta x}\,dx = \frac{2a\beta}{\beta^2+k^2} \end{aligned}$$

Name: ______________________________

PHYS2502 Mathematical Physics Quiz #9 31 Mar 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the norm of the vector $\underline{u} = \underline{\underline{A}}\,\underline{v}$ where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & i \\ -2 & i & 2 \\ 2 & -2 & 4 \end{bmatrix} \qquad \text{and} \qquad \underline{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

PHY2502 Mathematical Physics Quiz #9 Solution

Find the norm of the vector $\underline{u} = \underline{\underline{A}}\,\underline{v}$ where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & i \\ -2 & i & 2 \\ 2 & -2 & 4 \end{bmatrix} \qquad \text{and} \qquad \underline{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} 1 & 0 & i \\ -2 & i & 2 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2i \\ -2+2i+4 \\ 2-4+8 \end{bmatrix} = \begin{bmatrix} 1+2i \\ 2+2i \\ 6 \end{bmatrix}$$

$$\langle u|u\rangle = (1-2i)(1+2i) + (2-2i)(2+2i) + 6\cdot 6 = 1+4+4+4+36 = 49$$

$$\sqrt{\langle u|u\rangle} = 7$$

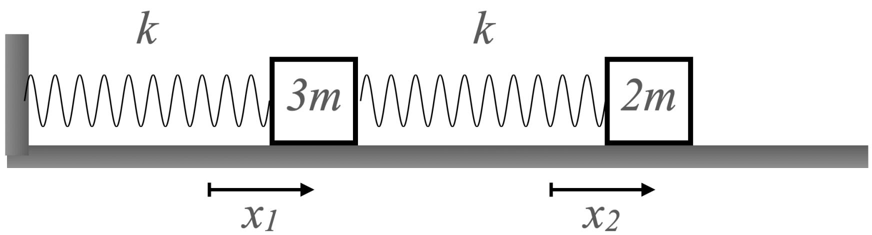
Name: ______________________________

PHYS2502 Mathematical Physics Quiz #10 7 Apr 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

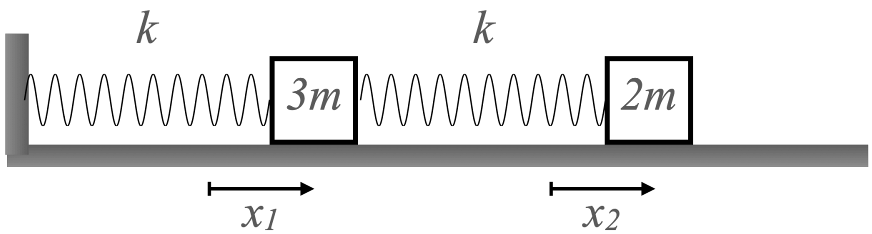
Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the eigenfrequencies of the coupled oscillator system below



PHY2502 Mathematical Physics Quiz #10 Solution

Find the eigenfrequencies of the coupled oscillator system below



You did this problem the hard way in HW#6 Problem#5. The equations of motion are

$$\begin{aligned} 3m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \\ 2m\ddot{x}_2 &= -k(x_2 - x_1) = kx_1 - kx_2 \end{aligned}$$

Inserting the standard ansatz, and writing using vectors and matrices, this becomes

$$\underline{\underline{\Omega}}\,\underline{x} = \frac{\omega^2}{\omega_0^2}\,\underline{x} \qquad \text{where} \qquad \underline{\underline{\Omega}} = \begin{bmatrix} 2/3 & -1/3 \\ -1/2 & 1/2 \end{bmatrix}$$

The characteristic equation for the eigenvalues $\lambda = \omega^2/\omega_0^2$ is

$$\left(\frac{2}{3} - \lambda\right)\left(\frac{1}{2} - \lambda\right) - \frac{1}{6} = \frac{1}{6}\left[(2 - 3\lambda)(1 - 2\lambda) - 1\right] = \frac{1}{6}\left[1 - 7\lambda + 6\lambda^2\right] = \frac{1}{6}\left[(1 - \lambda)(1 - 6\lambda)\right] = 0$$

so the eigenvalues are $\lambda = 1$ and $\lambda = 1/6$. The eigenfrequencies are

$$\omega^{(\lambda=1)} = \omega_0 \qquad \text{and} \qquad \omega^{(\lambda=1/6)} = \frac{1}{\sqrt{6}}\omega_0$$

Name: ________________________

PHYS2502 Mathematical Physics   Quiz #11   21 Apr 2022

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

This quiz is worth 20 points.

Find the first two nonzero terms in the Taylor expansion of $f(x) = \log(1 + x)$ about $x = 0$. (By "log" I mean the natural logarithm.)

PHY2502 Mathematical Physics Quiz #11 Solution

Find the first two nonzero terms in the Taylor expansion of $f(x) = \log(1+x)$ about $x = 0$. (By "log" I mean the natural logarithm.)

$$
\begin{aligned}
f(0) &= \log(1) = 0 \\
f'(0) &= \left.\frac{1}{1+x}\right|_{x=0} = 1 \\
f''(0) &= -\left.\frac{1}{(1+x)^2}\right|_{x=0} = -1 \\
f(x) &\approx f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 = x - \frac{1}{2}x^2
\end{aligned}
$$