PHYS2502 Mathematical Physics Homework #9 Due 22 Mar 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A function $u(x,t)$ satisfies the wave equation in one dimension x with velocity v. The initial conditions are $u(x,0) = p(x)$, for an arbitrary function $p(x)$, and $\dot{u}(x,0) = 0$. Show that the time development of the wave corresponds to the "splitting" of $p(x)$ into two pieces, one moving to the left and the other moving to the right, each being an exact copy of $p(x)$ but divided by two.

(2) A wave $u(x,t) = g(x+vt)$ moves to the left along a string on the positive x-axis.

(a) Assume the string is fixed at $x = 0$ so that it cannot move, that is $u(0,t) = 0$. Find the motion of the string for $x \geq 0$ for all times.

(b) Now ssume the string is free to move up and down at $x = 0$, and does so in a way that it is always horizontal, that is $\partial u(x,t)/\partial x|_{x=0} = 0$. Once again find the motion of the string for $x \geq 0$ for all times.

(3) Use a Fourier Sine decomposition to find the motion of a string that is fixed at $x = 0$ and $x = L$, and that starts from rest with an initial shape $u(x,0) = (2/L)^4 x^2 (x-L)^2$. (You'll want to use MATHEMATICA for this problem.) Follow the procedure in the notes for the lopsided triangle wave, and compare to your numerical solution from Lab #8. I encourage you do do this with an animation.

(4) This problem concerns the Fourier Transform and width of the Gaussian function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

(a) Find the Fourier Transform $A(k)$ of $f(x)$. The integral is not hard to do. Just complete the square in the exponent, and use what you know about Gaussian integrals.

(b) Calculate the "width" Δx (that is, the square root of the variance) of $f(x)$. (See the notes for details.) Again make use of what you know about Gaussian integrals.

(c) Next find the width Δk of $A(k)$.

(d) Determine the product $\Delta x \Delta k$. How does this compare to the same result for the triangle pulse that we derived in class?

(5) Show that the following relationships are consistent with the fundamental definition of the δ-function. You can make use of results derived in the notes, but you'll likely find it useful to emply integration by parts.

(a) $x\delta(x) = 0$

(b) $x\delta'(x) = -\delta(x)$

(c) $x^2\delta''(x) = 2\delta(x)$