PHYS2502 Mathematical Physics Homework #8 Due 15 Mar 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) For a scalar field $f(\vec{r})$ over some volume V in three dimensional space, prove that

$$\int_V \vec{\nabla} f \, dV = \oint_S f \, d\vec{S}$$

where S is the surface enclosing V. You can use the "cut the volume up into little bricks" approach we used in class, or you can try inventing a vector field $\vec{A} = f\vec{C}$ where $\vec{C}$ is some arbitrary constant vector, and then use a different surface theorem.

(2) Calculate the curl of the following vector field in both Cartesian coordinates and cylindrical polar coordinates:

$$\vec{A}(\vec{r}) = -\frac{\hat{\imath}y - \hat{\jmath}x}{x^2 + y^2} = \frac{\hat{\phi}}{r}$$

Now calculate directly the line integral $\oint \vec{A} \cdot d\vec{\ell}$ around a closed circle of radius R in the xy plane, centered at the origin. (I suggest you do this with the polar coordinate expression.) Can you reconcile these two seemingly inconsistent results?

(3) Calculate the divergence of the following vector field in both Cartesian coordinates and spherical polar coordinates:

$$\vec{A}(\vec{r}) = \frac{\hat{\imath}x + \hat{\jmath}y + \hat{k}z}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\hat{r}}{r^2}$$

Now calculate directly the surface integral $\oint \vec{A} \cdot d\vec{S}$ around a sphere of radius R, centered at the origin. (I suggest you do this with the polar coordinate expression.) Can you reconcile these two seemingly inconsistent results?

(4) Use the "Separation of Variables" approach to solve the partial differential equation

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

for the function $u(x,t)$ with the initial condition $u(x,0) = \sin(\pi x/2)$ and boundary conditions $u(2,t) = u(0,t) = 0$.

(5) Look for a solution to the Helmholtz Equation $\vec{\nabla}^2 f(r,\phi) + k^2 f(r,\phi) = 0$ in plane polar coordinates by writing $f(r,\phi) = R(r)\Phi(\phi)$. Now insist that $\Phi(\phi + 2\pi) = \Phi(\phi)$, that is $\Phi(\phi)$ must be "single valued", and show that solutions for $R(r)$ must be Bessel Functions $J_m(kr)$ of integer order m.