PHYS2502 Mathematical Physics Homework #7 Due 8 Mar 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Derive the unit vectors $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ in spherical polar coordinates in terms of the Cartesian unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the spherical polar coordinates r, θ, and ϕ. Your starting point should be the transformation equations that give you the Cartesian coordinates x, y, and z in terms of the spherical coordinates.

(2) The kinetic energy of a particle of mass m is $K = mv^2/2 = m\vec{v}\cdot\vec{v}/2$, where $\vec{v} = d\vec{r}/dt$ is the particle's velocity vector. Derive an expression for K in terms of the spherical coordinates r, θ, and ϕ and their rates of change $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ with respect to time. Simplify your result as much as possible. You can carry all this out with the chain rule and the same transformation equations you used above in (1), but there is also a much simpler way.

(3) Find an expression for the square of the magnitude of the cross product $|\vec{A}\times\vec{B}|^2$ in terms of the magnitudes of $\vec{A}$ and $\vec{B}$ and their dot product $\vec{A}\cdot\vec{B}$, in two different ways:

(a) Directly from the definitions of the magnitudes of $|\vec{A}\times\vec{B}|$ and $\vec{A}\cdot\vec{B}$.

(b) Using components and the summation convention, along with the relationship between the totally antisymmetric symbol and the Kronecker delta.

(4) Show that the gradient operator in spherical coordinates is given by

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

You can do this using the transformation equations and the chain rule for partial derivatives, but you don't need to do nearly that much work. Think of the gradient as a "directional derivative" as discussed in the notes and in class, and use the results of Problem (1) above to write down what is the change $d\vec{r}$ for each of the three orthogonal directions in spherical coordinates.

(5) Derive the Laplacian in plane polar coordinates, i.e. cylindrical coordinates with no z-component. That is, show that

$$\vec{\nabla}^2 f(r,\phi) = \vec{\nabla}\cdot\vec{\nabla}f(r,\phi) = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{r}\frac{\partial f}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial f}{\partial\phi}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial f}{\partial\phi^2}$$

Don't forget that you need to take into account that $\hat{r}$ and $\hat{\phi}$ depend explicitly on ϕ. Writing these unit vectors in terms of $\hat{\imath}$ and $\hat{\jmath}$ is probably the easiest way to do this.