PHYS2502 Mathematical Physics Homework #6 Due 22 Feb 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Consider the Euler Equations, with notation as described in the course notes.

(a) Use the Wronskian to show that the two solutions for $(\alpha-1)^2 > 4\beta^2$ are linearly independent for all $x > 0$. Recall that the Wronskian for two solutions $y_1(x)$ and $y_2(x)$ is $W[y_1(x), y_2(x)] = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

(b) For the case $(\alpha-1)^2 = 4\beta^2$, find a second solution and again show that the two solutions are linearly independent for all $x > 0$. *Hint: Try the approach that worked for a linear second order equation with constant coefficients.*

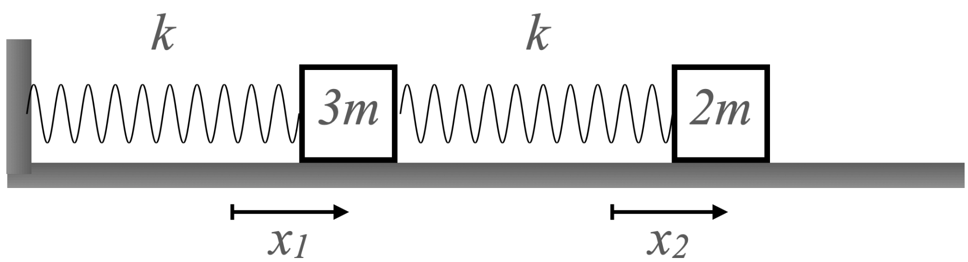
(2) Show that, for a Bessel Function $J_m(x)$ for integer order m, $J_{-m}(x) = (-1)^m J_m(x)$. You can use the Γ–Function to interpret $n!$ for $n < 0$. Explain why this means that $y(x) = c_1 J_m(x) + c_2 J_{-m}(x)$ cannot be the general solution to Bessel's Equation for $m \in \mathbb{Z}$.

(3) Show by explicit substitution that the Spherical Bessel Function $j_0(x) = \sin(x)/x$ of order zero, where $x = kr$, solves the $\ell = 0$ radial dependence of the Helmholtz Equation

$$r^2 R''(r) + 2rR'(r) + k^2 r^2 R(r) = 0$$

(4) Use Rodrigues' Formula to derive the first three Legendre Polynomials $P_0(x)$, $P_1(x)$, and $P_2(x)$, and compare to the results given in the course notes.

(5) Two masses $3m$ and $2m$ are connected to two identical springs as shown:



The masses are free to move horizontally and one spring is attached to a fixed wall.

(a) Write down Newton's Second Law for each of the two masses.

(b) Find the eigenfrequencies and describe the motion of the two eigenmodes.

(c) Write $x_1(t)$ and $x_2(t)$ in terms of four arbitrary constants a, b, c, and d.

(d) Make a plot of $x_1(t)$ and $x_2(t)$ subject to the initial conditions $x_1(0) = 1$, and $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$. (You can let MATHEMATICA solve for a, b, c, and d.)

(e) For the example in class, we found that the two combinations $x_\pm(t) = x_1(t) \pm x_2(t)$ oscillated with the two eigenfrequencies. What linear combinations $x_A(t)$ and $x_B(t)$ of $x_1(t)$ and $x_2(t)$ oscillate with the eigenfrequencies in this case? The answer should be clear from (c) above. Plot $x_A(t)$ and $x_B(t)$ and show they they oscillate with the correct frequencies.