PHYS2502 Mathematical Physics   Homework #3   Due 1 Feb 2022

This homework assignment is due at the start of class on the date shown. You can hand it in during class time, or email a PDF of your completed assignment to the instructor or grader, so that it arrives before the start of class.

(1) Find the first three nonzero terms of the Taylor expansions about $x = 0$ for $f(x) = \cosh(x)$ and $f(x) = \sinh(x)$. Make sketches of each of these two functions along with the approximations based on the first, second, and third terms. (You are welcome to work this problem in MATHEMATICA.)

(2) Two electric charges $\pm q$ lie at $z = \pm a/2$ on the z-axis.

(a) Find the magnitude of the electric field on the z-axis at distances far from the origin. Express your result in terms of the *electric dipole moment* $p = qa$. Compare how the field from an electric dipole falls with distance with that of an isolated electric charge.

(b) Repeat for a position on the x-axis, again, far from the origin. Indicate the direction of the electric field relative to that in (a).

(3) Consider the function $f(x) = x^n e^{-x}$. Find the value of x which maximizes $f(x)$, and sketch the function for some large value of n. Then write $x = e^{\log x}$ and write $f(x)$ in terms of $y \equiv x - n$. Expand the logarithm to second order in a Taylor series about $y = 0$ and show that $f(x)$ is a constant times a Gaussian function of y. Use this result, along with the definition of the Gamma function and Gaussian integrals to derive Stirling's Approximation, namely

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad \text{for} \qquad n \gg 1$$

(4) Derive expressions for $\cos(3x)$ and $\sin(3x)$ in terms of $\cos x$ and $\sin x$ by applying Euler's Formula.

(5) Reduce the following complex expressions into a simple complex (or purely real or purely imaginary) number of the form $z = x + iy$:

- i^i
- $\left[(1+i\sqrt{3})/(\sqrt{2}+i\sqrt{2})\right]^{50}$
- $\sinh(1 + i\pi/2)$
- $e^{2\tanh^{-1} i}$

For the last one, you'll need to come up with an expression for $\tanh^{-1}(x)$ in terms of the natural logarithm. (It's not hard.) Don't be afraid to write $\log e^\alpha = \alpha$ even if α is complex. You should be able to check all your answers using MATHEMATICA.