PHYS2502 Mathematical Physics Homework #1 Due 18 Jan 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Show that the product of pressure P and volume V has the dimension of energy. Use this and the ideal gas law $PV = NkT$, where N is the number of gas molecules and T is temperature in Kelvin, to find the SI units of Boltzmann's constant k.

(2) A simple harmonic oscillator is constructed from a mass m connected to a spring with stiffness k. The stiffness is determined by measuring the force from the spring when it is extended or compressed a certain distance, with the force being proportional to that distance.

(a) For classical oscillations with (position) amplitude A, use dimensional analysis to find the energy scale in terms of m, k, and A.

(b) For quantum mechanical oscillations, the amplitude is not well defined, but we expect the energy scale to depend also on $\hbar$, which has units of angular momentum. Find the energy scale in terms of m, k, and $\hbar$.

(3) The "Planck Length" ℓ_P is the distance at which gravity is unified with quantum mechanics and relativity. Find an expression for ℓ_P in terms of G, $\hbar$, and c. Evaluate it numerically and compare it to the size of the proton.

(4) In class we showed that the derivative with respect to x for $f(x) = x^n$, where n is a positive integer, is $f'(x) = nx^{n-1}$.

(a) Show that this relation also holds for $n = 0$.

(b) Use the definition of the derivative to show that this relation also holds when n is a negative integer.

(c) Use (a) and (b) to show that this relation still holds if $f(x) = x^{p/q}$ where p and q are integers, that is, when the exponent is a rational number. *Hint: Consider* $y^q = x^p$.

(d) Can you use all this to rationalize that the derivative of x^α is $\alpha x^{\alpha-1}$ for any $\alpha \in \mathbb{R}$?

(5) A particle moves in a circle in the (x, y) plane, centered on the origin. Find an expression that relates the velocity $v_x = dx/dt$ in the x-direction and the velocity $v_y = dy/dt$ in the y-direction to the position coordinates x and y. Draw a picture of a circle and indicate a few points on it that convince you that your answer is correct. Of course, you need to explain your reasoning.

(1) $[PV] = [\text{Force}]L^{-2}L^3 = MLT^{-2}]L^{-2}L^3 = ML^2T^{-2} = [\text{Energy}]$. From the ideal gas law, since N is dimensionless, kT must have the dimensions of energy. The SI unit of energy is Joule, so the units of k are Joule/K.

(2) First find $[k] = [\text{Force}]/[\text{Distance}] = MLT^{-2}L^{-1} = MT^{-2}$. (a) Write $\epsilon = m^x k^y A^z$, so

$$ML^2T^{-2} = M^x M^y T^{-2y} L^z = M^{x+y} T^{-2y} L^z$$

and $y = 1$, $z = 2$, and $x = 0$. That is, the energy scale is $\epsilon = kA^2$. You'll recall that the potential energy of a harmonic oscillator with amplitude A is $kA^2/2$ so the scale is the correct energy to within a factor of two. (b) Write $\epsilon = m^x k^y \hbar^z$, so

$$ML^2T^{-2} = M^x M^y T^{-2y} L^{2z} M^z T^{-z} = M^{x+y+z} T^{-2y-z} L^{2z}$$

and $z = 1$, $2y + z = 2y + 1 = 2$ so $y = 1/2$, and $x + y + z = x + 3/2 = 1$ so $x = -1/2$. Therefore $\epsilon = \hbar(k/m)^{1/2}$. In quantum mechanics you will learn that the energy spacing in the harmonic oscillator is indeed $\hbar\omega$ where $\omega = (k/m)^{1/2}$.

(3) Newton's gravitational law gives $[G] = L^3M^{-1}T^{-2}$, so write $\ell_P = G^x\hbar^y c^z$ which gives

$$L = L^{3x}M^{-x}T^{-2x}L^{2y}M^yT^{-y}L^zT^{-z} = L^{3x+2y+z}M^{-x+y}T^{-2x-y-z}$$

Therefore $x = y$, $2x + y + z = 3x + z = 0$, and $3x + 2y + z = 5x + z = 1$. Subtract the last two equations to get $x = 1/2 = y$, so $z = -3x = -3/2$ and $\ell_P = (G\hbar/c^3)^{1/2}$. This works out to be 1.6×10^{-35} m, 20 orders of magnitude smaller than the proton.

(4) (a) For $n = 0$, $f(x) = 1$ which does not change with x, so $f'(x) = 0$. (b) With $m = -n$,

$$f'(x) = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}\right] = -\lim_{\Delta x\to 0}\frac{mx^{m-1}\Delta x + \cdots}{x^m(x+\Delta x)^m} = -mx^{-m-1} = nx^{n-1}$$

(c) Put $y = f(x)$ so that $y^q = x^p$. Then $qy^{q-1}dy = px^{p-1}dx$ and

$$f'(x) = \frac{dy}{dx} = \frac{p}{q}x^{p-1}y^{1-q} = \frac{p}{q}x^{p-1}\left(x^{p/q}\right)^{1-q} = \frac{p}{q}x^{p-1}x^{p/q-p} = \frac{p}{q}x^{p/q-1}$$

(d) Any real number is infinitesimally close to a rational number on either side, so, sure, it makes sense that this rule holds for any real number exponent.

(5) The equation of a circle is $x^2 + y^2 = R^2$ so $2x\,dx + 2y\,dy = 0$. Dividing through by $2dt$ tells us that $xv_x = -yv_y$. At the point $(x, y) = (R, 0)$ the x-component of the velocity is indeed zero, and at the point $(x, y) = (0, R)$ the y-component of the velocity is also zero. Most telling, though is when the particle is at 45° in the first quadrant, that is $(x, y) = (R/\sqrt{2}, R/\sqrt{2})$. Here, v_x and v_y are equal in magnitude but opposite in sign, and the relation still holds. Similarly in the other three quadrants.

PHYS2502 Mathematical Physics Homework #2 Due 25 Jan 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) The energy of a simple harmonic oscillator made of a mass m and a spring with stiffness constant k imoving in one dimension x is $E = mv^2/2 + kx^2/2$, where $v = dx/dt$.

(a) Take the derivative of the right side, along with Newton's Second Law and Hooke's Law, to show that the energy does not change with time.

(b) Integrate over the quarter of a period where both v and x are positive, and derive an expression for the period T in terms of k and m. The integral is easy to carry out using a change of variables involving a circular function.

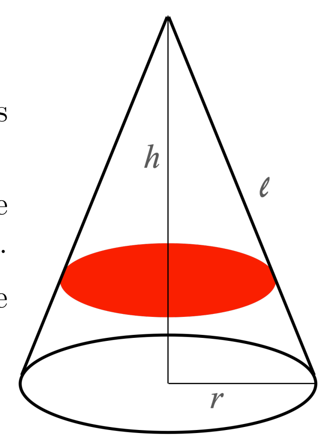
(2) A dam in the shape of an inverted triangle blocks a river valley, forming a lake of depth D and width W. Taking the water pressure $p(y) = \rho g y$ at depth y from the surface of the lake, find the total force acting on the dam. Check that your result is dimensionally correct. Calculate the force on the Hoover Dam ($W = 200$ m) from Lake Mead ($D = 160$ m). Express your result in tons of force.

(3) Find the derivative of $\tan x \equiv \sin x / \cos x$ with respect to x. Then use the change of variables $ax = \tan u$ to evaluate the integral

$$\int_0^\infty \frac{dx}{1 + a^2 x^2}$$

You might want to check your answer using MATHEMATICA.

(4) Show that $f(x) = \int_1^x (1/t)dt$ has the property $f(ab) = f(a) + f(b)$ using an appropriate change of integration variables. Hence show that $f(a^n) = nf(a)$ for $n \in \mathbb{Z}^+$.



(5) Consider a right circular cone of height h and base radius r, as shown on the right. Let ℓ be the slant height of the cone.

(a) Find the volume V in terms of h and r by adding up the volume of a bunch of thin circular disks, one of which is shown in red.

(b) Now find the ratio h/r that maximizes the volume of the cone for a fixed slang length ℓ.

PHY2502 Mathematical Physics Homework #2 Solutions

(1) Newton's Second Law is $F = ma = mdv/dt$ and Hooke's Law is $F = -kx$ so

$$\frac{dE}{dt} = \frac{1}{2}m(2v)\frac{dv}{dt} + \frac{1}{2}k(2x)\frac{dx}{dt} = v\,m\frac{dv}{dt} + kx\,v = v\,F + (-F)\,v = 0$$

The time $T/4$ is the integral of t from $x = 0$ to $x = (2E/k)^{1/2}$, so

$$T = 4\int_{x=0}^{(2E/k)^{1/2}} dt = 4\int_0^{(2E/k)^{1/2}} \frac{dx}{v} = 4\int_0^{(2E/k)^{1/2}} \frac{dx}{\sqrt{(2/m)(E - kx^2/2)}}$$

Make the change of variables $x = (2E/k)^{1/2}\sin u$ in which case u runs from $u = 0$ to $u = \pi/2$. Also $E - kx^2/2 = E(1 - \sin^2 u) = E\cos^2 u$, and $dx = (2E/k)^{1/2}\cos u\,du$. Therefore

$$T = 4\sqrt{\frac{m}{2}}\sqrt{\frac{2E}{k}}\frac{1}{\sqrt{E}}\int_0^{\pi/2}\frac{\cos u\,du}{\cos u} = 4\sqrt{\frac{m}{k}}\frac{\pi}{2} = \frac{2\pi}{\sqrt{k/m}}$$

(2) The force dF on a horizontal strip of the dam is given by the product of the pressure and the area of the strip. The area of the strip is $w(y)dy$ where $y = 0$ at the lake's surface and $y = D$ at the bottom of the lake. We need $w(0) = W$ and $w(D) = 0$ where $w(y) = ay + b$ since the width varies linearly with depth. Therefore $b = W$ and $aD + b = aD + W = 0$ so $a = -W/D$ giving $w(y) = W(1 - y/D)$. Then

$$F = \int_0^D dF = \int_0^D p(y)w(y)dy = \rho g W\int_0^D\left(y - \frac{y^2}{D}\right)dy = \rho g W\left(\frac{D^2}{2} - \frac{1}{D}\frac{D^3}{3}\right) = \frac{\rho g W D^2}{6}$$

It is easy to see that this is dimensionally correct, since ρ has dimensions of mass per unit volume and WD^2 has dimensions of volume, so $\rho g W D^2$ has dimensions of mass times acceleration, which is force.

In SI units, $\rho = 10^3$ kg/m^3, $g = 9.8$ m/s^2, and $WD^2 = 200 \times 160^2 = 5.1 \times 10^6$ m^3, so the total force is $10^3 \times 9.8 \times 5.1 \times 10^6/6 = 8.4 \times 10^9$ N$=2 \times 10^9$ pounds or a million tons.

(3) To find the derivative of $\tan x$, just use the chain rule, that is

$$\frac{d}{dx}\frac{\sin x}{\cos x} = \left[\frac{\cos x}{\cos x} - \frac{\sin x\,(-\sin x)}{\cos^2 x}\right] = \frac{1}{\cos^2 x}$$

Now $ax = \tan u$ means that $x = 0$ corresponds to $u = 0$ and $x = \infty$ to $u = \pi/2$. Also

$$\frac{1}{1+a^2x^2} = \frac{1}{1+\tan^2 u} = \frac{\cos^2 u}{\cos^2 u + \sin^2 u} = \cos^2 u$$

The integral then becomes simple. We have $a\,dx = du/\cos^2 u$ and

$$\int_0^\infty \frac{1}{1+a^2x^2}\,dx = \frac{1}{a}\int_0^{\pi/2} \cos^2 u \frac{du}{\cos^2 u} = \frac{\pi}{2a}$$

(4) Use $u = t/b$, then split the integral and use $f(1/a) = -f(a)$ to get

$$f(ab) = \int_1^{ab} \frac{1}{t}dt = \int_{1/b}^{a} \frac{1}{bu}bdu = \int_1^a \frac{1}{u}du + \int_{1/b}^{1} \frac{1}{u}du = f(a) - f(1/b) = f(a) + f(b)$$

Obviously, then, $f(a^2) = 2f(a)$ and so on for $f(a^n)$ for any positive integer n.

(5) Let y measure the vertical distance from the base to the tip. Then the radius $\rho(y)$ of a circular disk is $\rho(y) = r - (r/h)y = r(1 - y/h)$. The volume of a circular disk is $dV = \pi\rho^2 dy$ so the volume of the cone is

$$V = \int_0^h dV = \int_0^h \pi\rho^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{h}y + \frac{1}{h^2}y^2\right) dy = \pi r^2 \left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}\pi r^2 h$$

To maximize the volume for fixed ℓ we write $V = \pi(\ell^2 - h^2)h/3$ and then find the derivative with respect to h. We have

$$\frac{dV}{dh} = \frac{\pi}{3}\frac{d}{dh}(\ell^2 - h^2)h = \frac{\pi}{3}(\ell^2 - 3h^2) = 0$$

Therefore the volume is a maximum when $h = \ell/\sqrt{3}$, or

$$\frac{h}{r} = \frac{h}{\sqrt{\ell^2 - h^2}} = \frac{1}{\sqrt{\ell^2/h^2 - 1}} = \frac{1}{\sqrt{3-1}} = \frac{1}{\sqrt{2}}$$

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PHYS2502 Mathematical Physics Homework #3 Due 1 Feb 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Find the first three nonzero terms of the Taylor expansions about $x = 0$ for $f(x) = \cosh(x)$ and $f(x) = \sinh(x)$. Make sketches of each of these two functions along with the approximations based on the first, second, and third terms. (You are welcome to work this problem in Mathematica.)

(2) Two electric charges $\pm q$ lie at $z = \pm a/2$ on the z-axis.

(a) Find the magnitude of the electric field on the z-axis at distances far from the origin. Express your result in terms of the *electric dipole moment* $p = qa$. Compare how the field from an electric dipole falls with distance with that of an isolated electric charge.

(b) Repeat for a position on the x-axis, again, far from the origin. Indicate the direction of the electric field relative to that in (a).

(3) Consider the function $f(x) = x^n e^{-x}$. Find the value of x which maximizes $f(x)$, and sketch the function for some large value of n. Then write $x = e^{\log x}$ and write $f(x)$ in terms of $y \equiv x - n$. Expand the logarithm to second order in a Taylor series about $y = 0$ and show that $f(x)$ is a constant times a Gaussian function of y. Use this result, along with the definition of the Gamma function and Gaussian integrals to derive Stirling's Approximation, namely

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad \text{for} \qquad n \gg 1$$

(4) Derive expressions for $\cos(3x)$ and $\sin(3x)$ in terms of $\cos x$ and $\sin x$ by applying Euler's Formula.

(5) Reduce the following complex expressions into a simple complex (or purely real or purely imaginary) number of the form $z = x + iy$:

- i^i
- $\left[(1 + i\sqrt{3})/(\sqrt{2} + i\sqrt{2})\right]^{50}$
- $\sinh(1 + i\pi/2)$
- $e^{2\tanh^{-1} i}$

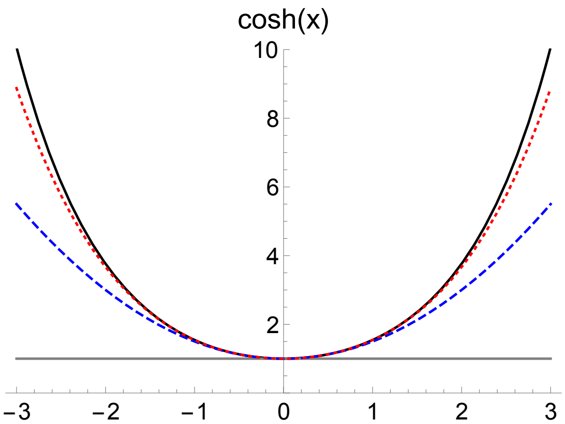
For the last one, you'll need to come up with an expression for $\tanh^{-1}(x)$ in terms of the natural logarithm. (It's not hard.) Don't be afraid to write $\log e^\alpha = \alpha$ even if α is complex. You should be able to check all your answers using Mathematica.

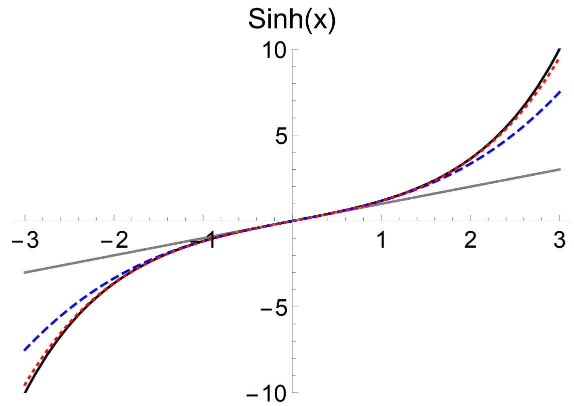
PHY2502 Mathematical Physics Homework #3 Solutions

(1) The derivatives of $\cosh(x)$ and $\sinh(x)$ are each other, with no minus signs, so their Taylor expansions are

$$\cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \qquad \text{and} \qquad \sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

The plots follow. See the accompanying MATHEMATICA notebook for the details.





(2) (a) Consider the positive z-axis, in which case the distance from $(\pm)q$ is $z \mp a/2$. The magnitude of the electric field is

$$E = +\frac{q}{(z-a/2)^2} - \frac{q}{(z+a/2)^2} = \frac{q}{z^2}\left[\frac{1}{(1-a/2z)^2} - \frac{1}{(1+a/2z)^2}\right]$$

Since $z \gg a$, $a/2z \ll 1$ and we take $(1 \mp a/2z)^{-2} = 1 \pm 2(a/2z) = 1 \pm a/z$. So,

$$E = \frac{q}{z^2}\left[\left(1+\frac{a}{z}\right) - \left(1-\frac{a}{z}\right)\right] = \frac{2qa}{z^3} = \frac{2p}{z^3}$$

The field from the dipole falls like distance cubed, rather than distance squared.

(b) The field along the x-axis points downward, that is, in the $-z$ direction, because the horizontal components from the two charges cancel. In this case, the magnitude of the electric field of each charge must therefore be multiplied by cosine of the angle between the z-axis and the direction from the charge to the field point on the x-axis. The distance from each charge to the field point is $r = (x^2 + a^2/4)^{1/2}$ so the cosine of the angle is $(a/2)/r$. The fields from the two charges add, so with a negative sign to indicat the downward direction,

$$E = -2\frac{q}{r^2}\frac{a}{2r} = -\frac{p}{(x^2+a^2/4)^{3/2}} \to -\frac{p}{x^3}$$

for $x \gg a$. Once again, the field falls like the distance cubed, but points downward instead of upward.

(3) First just take the derivative of $f(x)$ to find the maximum:

$$\frac{d}{dx}x^n e^{-x} = nx^{n-1}e^{-x} - x^n e^{-x} = x^{n-1}e^{-x}(n-x) = 0$$

so the maximum is reached at $x = n$. For a large value of n, the factor x^n rises rapidly with a large second derivative until the peak at $x = n$, at which point the exponential controls and the function falls rapidly. A sketch should make it look like a narrow Gaussian, more or less, peaking at $y = x - n = 0$. The second order expansion of the logarithm is

$$\log(1+t) = t - \frac{1}{2}t^2$$

so we can write $f(x)$ for large n in the form

$$\begin{aligned} x^n e^{-x} &= e^{n\log x}e^{-x} = \exp[n\log(n+y) - n - y] = \exp[n\log n + n\log\left(1+\frac{y}{n}\right) - n - y] \\ &= \exp[n\log n + n\left(\frac{y}{n} - \frac{y^2}{2n^2}\right) - n - y] = e^{n\log n - n}e^{-y^2/2n} = n^n e^{-n}e^{-y^2/2n} \end{aligned}$$

Now $n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x}$. Since the integrand is very narrow for $n \gg 1$ and it peaks for a large $x = n$, we might as well integrate over the entire real axis. Using

$$\int_{-\infty}^{\infty} e^{-y^2/2n}dy = \sqrt{2\pi n}$$

we arrive at

$$n! \approx n^n e^{-n}\sqrt{2\pi n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

(4) We just need to write $e^{3ix} = (e^{ix})^3$ and get

$$\begin{aligned} \cos(3x) + i\sin(3x) &= (\cos x + i\sin x)^3 \\ &= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x \\ &= \cos^3 x - 3\cos x\sin^2 x + i(3\cos^2 x\sin x - \sin^3 x) \\ \text{so} \qquad \cos(3x) &= \cos^3 x - 3\cos x\sin^2 x \\ \text{and} \qquad \sin(3x) &= 3\cos^2 x\sin x - \sin^3 x \end{aligned}$$

(5) The key to all of these is to make use of Euler's formula.

$$\begin{aligned}
i^i &= \left(e^{i\pi/2}\right)^i = e^{-\pi/2} \\
\left[\frac{1+i\sqrt{3}}{\sqrt{2}+i\sqrt{2}}\right]^{50} &= \left[\frac{2}{2}\right]^{50}\left[\frac{\exp(i\pi/3)}{\exp(i\pi/4}\right]^{50} = \left[e^{i\pi/12}\right]^{50} = e^{4i\pi}e^{i\pi/6} = e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} \\
\sinh(1+i\pi/2) &= \sinh(1)\cosh(i\pi/2) + \sinh(i\pi/2)\cosh(1) \\
&= i\sinh(1)\cos(\pi/2) + i\sin(\pi/2)\cosh(1) = i\cosh(1)
\end{aligned}$$

To get an expression for $y = \tanh^{-1}(x)$, write $x = \tanh(y)$ and solve for y.

$$\begin{aligned}
x &= \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1} \\
xe^{2y} + x &= e^{2y} - 1 \\
e^{2y}(1-x) &= 1 + x \\
\text{so} \qquad y &= \frac{1}{2}\log\frac{1+x}{1-x} = \tanh^{-1}(x)
\end{aligned}$$

Now we can calculate

$$\tanh(i) = \frac{1}{2}\log\frac{1+i}{1-i} = \frac{1}{2}\log\frac{\sqrt{2}\exp(i\pi/4)}{\sqrt{2}\exp(-i\pi/4)} = \frac{1}{2}\log e^{i\pi/2} = \frac{1}{2}\log i$$

Therefore $e^{2\tanh^{-1} i} = e^{\log i} = i$.

PHYS2502 Mathematical Physics   Homework #4   Due 8 Feb 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Find the solution of the differential equation $dx/dt = x^2$ where x is position and t is time, and where $x(0) = a$ where $a > 0$. For what range of times is your solution valid? *Careful! This is a trick question.*

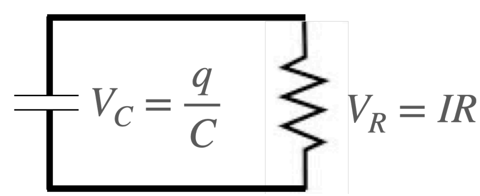
(2) The pressure $P(T)$ along a liquid-gas phase boundary on a pressure vs temperature (T) diagram is the solution to the differential equation

$$\frac{dP}{P} = k\frac{dT}{T^2}$$

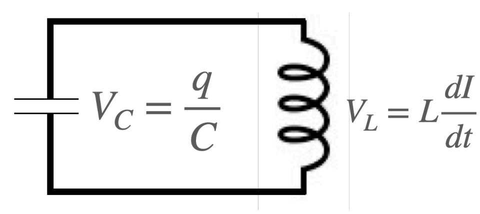
where k is a a constant. If the pressure is P_0 at a temperature T_0, find the function $P(T)$.

(3) An object of mass m falls from rest some distance above the Earth's surface. It is subject to a drag force av^2 proportional to its velocity. Find its velocity $v(t)$, and check that your answer is dimensionally correct. Then check that you get the correct behavior for both short and long times. I suggest that, as we did in class, that you choose a coordinate system where "up" is positive. (If you carry out the necessary integrals using MATHEMATICA, then please also submit a PDF of your executed notebook.)

(4) The diagram at the right shows a capacitor C connected in series with a resistor R. The potential difference across the capacitor is $V_C = q/C$ where q is the charge stored on the capacitor. The potential difference across the resistor is $V_R = IR$ where $I = dq/dt$ is the current through the resistor. If the initial charge on the capacitor is q_0, find $q(t)$ as a function of time.



(5) The diagram at the right shows a capacitor C connected in series with an inductor L. The potential difference across the capacitor is $V_C = q/C$ where q is the charge stored on the capacitor. The potential difference across the inductor is $V_L = LdI/dt$ where $I = dq/dt$ is the current through the inductor. If the initial charge on the capacitor is q_0, and the initial current is zero, find $q(t)$ as a function of time.



PHY2502 Mathematical Physics Homework #4 Solutions

(1) This is a separable equation. Integrate $dx/x^2 = dt$ to get $-1/x = t + C$. The initial condition gives $-1/a = C$ so the function is $x(t) = -1/(t - 1/a) = a/(1 - at)$. As a simple check, $dx/dt = -a/(1-at)^2(-a) = a^2/(1-at)^2 = x^2$. now since $x(0) > 0$ and the solution becomes negative for $t > 1/a$, and blows up at $t = 1/a$, the solution is only valid for $0 \leq t < 1/a$. Admittedly, this problem appears to be rather "unphysical."

(2) Just integrate both sides to get $\log P = -k/T + C$ and write $\log P_0 = -k/T_0 + C$. Therefore $P = e^C e^{-k/T}$ and $P_0 = e^C e^{-k/T_0}$ so the solution is $P(T) = P_0 e^{k/T_0} e^{-k/T}$ This is more neatly written as $P(T) = P_0 e^{-k(1/T - 1/T_0)} = P_0 e^{k(T-T_0)/TT_0}$.

(3) The net force on the mass is $-mg + av^2$ where we note that the drag force is always positive when the mass is falling. Newton's Second Law says $mdv/dt = -mg + av^2$ or

$$\begin{aligned}
\int_0^t (-g)\,dt &= \int_0^v \frac{dv}{1 - av^2/mg} = \frac{1}{2}\int_0^v \left[\frac{dv}{1 - \sqrt{a/mg}v} + \frac{dv}{1 + \sqrt{a/mg}v}\right] \\
-2gt &= -\sqrt{\frac{mg}{a}} \log\left[1 - \sqrt{\frac{a}{mg}}v\right] + \sqrt{\frac{mg}{a}} \log\left[1 + \sqrt{\frac{a}{mg}}v\right] \\
e^{-2\sqrt{ag/m}t} &= \frac{1 + \sqrt{a/mg}v}{1 - \sqrt{a/mg}v} \qquad \text{so} \qquad v(t) = -\sqrt{\frac{mg}{a}} \tanh\left(\sqrt{\frac{ag}{m}}t\right)
\end{aligned}$$

First check dimensions. Start with $[a]L^2T^{-2} = MLT^{-2}$ so $[a] = ML^{-1}$. Then

$$\left[\sqrt{\frac{mg}{a}}\right] = \left(M \cdot LT^{-2} \cdot LM^{-1}\right)^{1/2} = LT^{-1} \quad \text{and} \quad \left[\sqrt{\frac{ag}{m}}\right] = \left(ML^{-1} \cdot LT^{-2} \cdot M^{-1}\right)^{1/2} = T^{-1}$$

both of which are correct. For $t \ll \sqrt{m/ag}$, $\tanh(\sqrt{ag/m}t) \to \sqrt{ag/m}t$ and so $v(t) \to -gt$, which is correct. For $t \gg \sqrt{m/ag}$, $\tanh(\sqrt{ag/m}t) \to 1$ and $v(t) \to -\sqrt{mg/a}$, the (constant) terminal velocity expected since $mg = av^2_{\text{term}}$.

(4) The potential differences must sum to zero over the loop, so

$$\frac{q}{C} + IR = \frac{q}{C} + \frac{dq}{dt}R = 0 \qquad \text{so} \qquad \frac{dq}{dt} = -\frac{q}{RC} \qquad \text{so} \qquad q(t) = q_0 e^{-t/\tau}$$

The capacitor discharges exponentially over time with time constant $\tau = RC$.

(5) The potential differences must sum to zero over the loop, so

$$\frac{q}{C} + L\frac{dI}{dt} = \frac{q}{C} + L\frac{d^2q}{dt^2} = 0 \qquad \text{so} \qquad \frac{d^2q}{dt^2} = -\frac{q}{LC} \qquad \text{so} \qquad q(t) = q_0 \cos\omega t$$

The charge oscillates over time with frequency $\omega = 1/\sqrt{LC}$.

PHYS2502 Mathematical Physics Homework #5 Due 15 Feb 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Make some sketches of the motion of $x(t)$ for the damped oscillator, similar to Figure 3.2 and 3.3 in the notes. Plot against time in units of the fundamental period of the undamped oscillator. You may carry out the calculations and make the plots using MATHEMATICA or some other application. Plot the following cases:

(a) $\beta = 0.05\omega_0$, $x_0 = 5$, $v_0 = 2\omega_0$

(b) $\beta = 1.5\omega_0$, $x_0 = 1$, $v_0 = -2\omega_0$

(c) $\beta = \omega_0$, $x_0 = 1$, $v_0 = -2\omega_0$

(2) Reproduce the plots in Figure 3.5 of the notes, that is $x(t)$ for a forced damped oscillator with $\beta = 0.05\omega_0$, $\gamma = 10$, and with initial conditions $x(0) = \dot{x}(0) = 0$. The three plots are for $\omega = 0.5\omega_0$, $\omega = \omega_0$, and $\omega = 1.5\omega_0$. Time is plotted in units of the fundamental period of the undamped oscillator. You only need to make the three plots, not necessarily on the same set of axes, but don't be afraid to try making the plot this way. If you want to be ambitious, consider using the Manipulate function in MATHEMATICA to see how the plot behaves if you let ω be adjustable on a sliding scale. (This is a nice demonstration of resonance.)

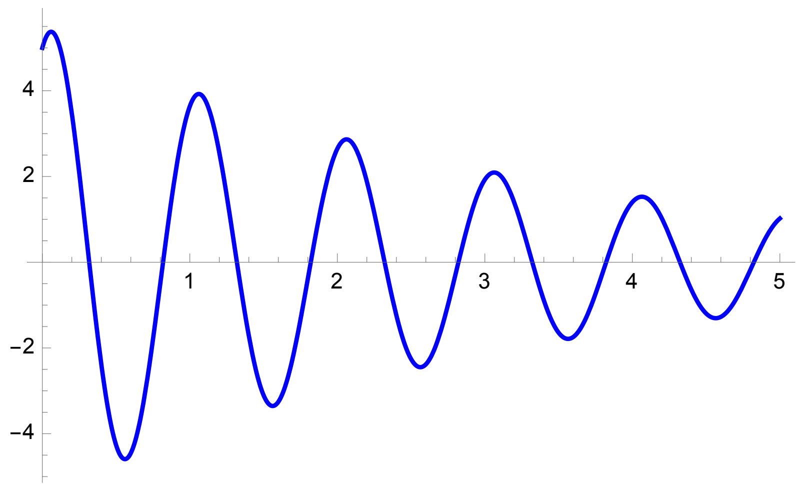
(3) In the terminology we used in class and in the notes, we saw that if $\beta > \omega_0$ ("over damping"), then the solution to the damped oscillator is the sum of two exponential functions, no matter how close β is to ω_0. However, if $\beta = \omega_0$ ("critical damping"), the solution magically turns into a single exponential dependence. Write $\omega_0^2 = \beta^2(1 - \epsilon^2)$ and show that for $\epsilon \ll 1$ the over damped solution turns into the critically damped solution.

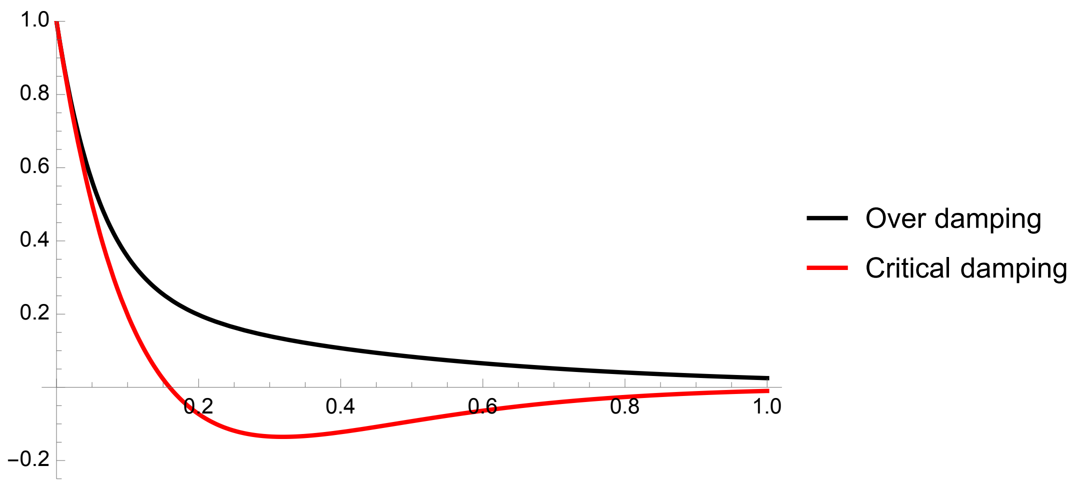
(4) Use the series approach to find the solutions for $y'' = y(x)$ and show that the result is the same as the series expansion for $y(x) = c_1 \cosh(kx) + c_2 \sinh(kx)$. How would you define constants a_1 and a_2 in terms of c_1 and c_2 so that the solution is $y(x) = a_1 e^x + a_2 e^{-x}$?

(5) Find general solution to $y''(x) = xy(x)$, which I will call "Keen's Equation", using series approach. Show that there can be no term in the series proportional to x^2, and that the recursion relations relate every third term of the expansion. Separate the two solutions you find for Keen's Equation, and explicitly indicate the constants of integration. Write out the first ten or so nonzero terms of each of the two solutions, and plot them. (Don't go too far in $\pm x$ so that you over run the range afforded by the number of terms you calculated in the expansion!) Note the difference in behavior for $x < 0$ and $x > 0$.

PHY2502 Mathematical Physics Homework #5 Solutions

(1) See the Mathematica notebook for details. Following are the plots:





(2) See the Mathematica notebook, including using Manipulate. It seems that using c1 and c2 for the arbitrary constants in the homogeneous solution is problematic. Note that I used the "cos" and "sin" version of the homogeneous solution.

(3) The over damped solution for the damped harmonic oscillator becomes

$$\begin{aligned} x(t) &= e^{-\beta t}\left[c_1 e^{-\sqrt{\beta^2-\omega_0^2}t} + c_2 e^{+\sqrt{\beta^2-\omega_0^2}t}\right] = e^{-\beta t}\left[c_1 e^{-\beta\epsilon t} + c_2 e^{\beta\epsilon t}\right] \\ &\approx e^{-\beta t}\left[c_1(1-\beta\epsilon t) + c_2(1+\beta\epsilon t)\right] \\ &= e^{-\beta t}\left[(c_1+c_2) - (c_1-c_2)\beta\epsilon t\right] \end{aligned}$$

Redefining $c_1 + c_2 \to c_1$ and $-(c_1 - c_2)\beta\epsilon \to c_2$ recovers the critically damped solution. It does not matter that the "new" c_2 multiplies ϵ because we can just make $c_1 - c_2$ as large as we need to.

(4) Writing $y(x) = \sum_{n=0}^{\infty} a_n x^n$ we have $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n$. Realizing that $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$, we make the substitution $m = n-2$, and then replace m with n in the summation. This gives us

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] x^n = 0 \qquad \text{so} \qquad a_{n+2} = \frac{1}{(n+2)(n+1)} a_n$$

Writing $c_1 = a_0$ and $c_2 = a_1$, we start with a_0 and get

$$a_2 = \frac{1}{2 \cdot 1} c_1 \qquad a_4 = \frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} c_1 \qquad a_6 = \frac{1}{6 \cdot 5} a_4 = \frac{1}{6!} c_1$$

and so on. If instead we start with a_1, we get

$$a_3 = \frac{1}{3 \cdot 2} c_2 \qquad a_5 = \frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} c_2 \qquad a_7 = \frac{1}{7 \cdot 6} a_5 = \frac{1}{7!} c_2$$

In other words $a_n = c_1/n!$ if n is even, and $a_n = c_2/n!$ if n is odd. That is

$$y(x) = c_1 \left[1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots\right] + c_2 \left[x + \frac{1}{3!}x^3 + \frac{1}{5}x^5 + \frac{1}{7!}x^7 + \cdots\right]$$

These are the Taylor expansions for $\cosh(x)$ and $\sinh(x)$ that we found in Problem 1 of Homework #3. Therefore

$$y(x) = c_1 \cosh(x) + c_2 \sinh(x)$$

If we write $c_1 = a_1 + a_2$ and $c_2 = a_1 - a_2$, then the solution becomes

$$\begin{aligned} y(x) &= a_1 \left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots\right] \\ &\quad + a_2 \left[1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5}x^5 + \frac{1}{6!}x^6 - \frac{1}{7!}x^7 + \cdots\right] \\ &= a_1 e^x + a_2 e^{-x} \end{aligned}$$

(5) Actually, this is the Airy Equation. The series solutions are derived in Boyce and DiPrima 6e Section 5.2. It's a good exercise for series solutions, though, so let's move through it. We start as usual with $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and write

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2}$$

where we pull out the first term because Keen's Equation is going to push up the power of the series on the right side of the equation. That is, Keen's Equation becomes, with $n = m+3$,

$$2a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + \sum_{m=0}^{\infty} (m+3)(m+2)a_{m+3} x^{m+1} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

So, after switching back $m \to n$, we get

$$2a_2 + \sum_{n=0}^{\infty} \left[n+3)(n+2)a_{n+3} - a_n\right] x^{n+1} = 0$$

This immediately tells us that $a_2 = 0$ so there is no term in the Keen Function proportional to x^2. The recursion relation is

$$a_{n+3} = \frac{1}{(n+3)(n+2)} a_n$$

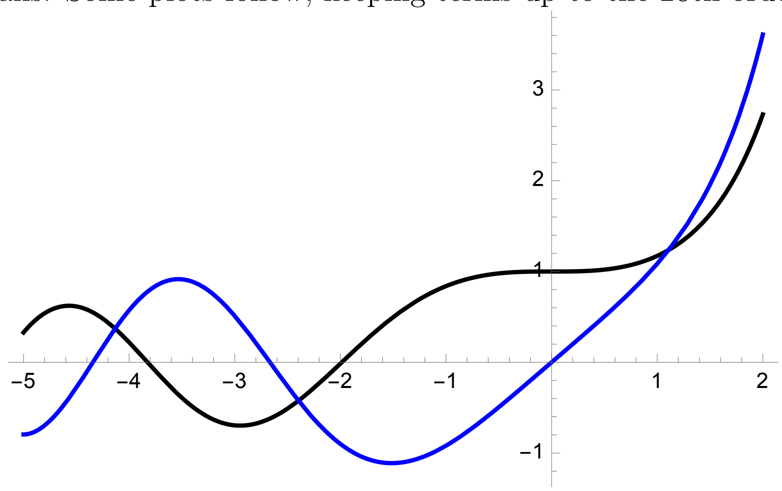
and we get one series for $a_0 = c_1$ and another series for $a_1 = c_2$. These have coefficients

$$a_3 = \frac{c_1}{3 \cdot 2} \qquad a_6 = \frac{a_3}{6 \cdot 5} = \frac{c_1}{6 \cdot 5 \cdot 3 \cdot 2} \qquad a_9 = \frac{a_6}{9 \cdot 8} = \frac{c_1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} \qquad \cdots$$

and

$$a_4 = \frac{c_2}{4 \cdot 3} \qquad a_7 = \frac{a_4}{7 \cdot 6} = \frac{c_2}{7 \cdot 6 \cdot 4 \cdot 3} \qquad a_{10} = \frac{a_7}{10 \cdot 9} = \frac{c_2}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} \qquad \cdots$$

A general formula for these coefficients is not obvious to me, but we can at least use the first four nonzero terms given here to see what the functions look like. See the Mathematica notebook for details. Some plots follow, keeping terms up to the 28th order polynomial.



See also Figure 5.2.3 in B&P6e. The point is that the solution oscillates for $x < 0$, but grows for $x > 0$. (I am not sure what is the linear combination of these two solutions that are called the Airy Functions, but the regular Airy Function falls to zero for $x > 0$.)

PHYS2502 Mathematical Physics Homework #6 Due 22 Feb 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Consider the Euler Equations, with notation as described in the course notes.

(a) Use the Wronskian to show that the two solutions for $(\alpha - 1)^2 > 4\beta^2$ are linearly independent for all $x > 0$. Recall that the Wronskian for two solutions $y_1(x)$ and $y_2(x)$ is $W[y_1(x), y_2(x)] = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

(b) For the case $(\alpha - 1)^2 = 4\beta^2$, find a second solution and again show that the two solutions are linearly independent for all $x > 0$. *Hint: Try the approach that worked for a linear second order equation with constant coefficients.*

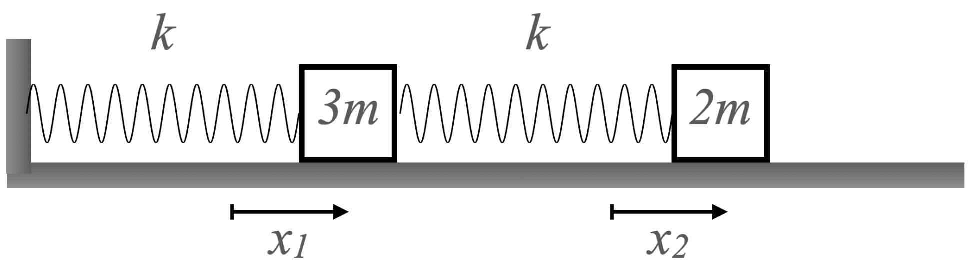
(2) Show that, for a Bessel Function $J_m(x)$ for integer order m, $J_{-m}(x) = (-1)^m J_m(x)$. You can use the Γ–Function to interpret $n!$ for $n < 0$. Explain why this means that $y(x) = c_1 J_m(x) + c_2 J_{-m}(x)$ cannot be the general solution to Bessel's Equation for $m \in \mathbb{Z}$.

(3) Show by explicit substitution that the Spherical Bessel Function $j_0(x) = \sin(x)/x$ of order zero, where $x = kr$, solves the $\ell = 0$ radial dependence of the Helmholtz Equation

$$r^2 R''(r) + 2rR'(r) + k^2 r^2 R(r) = 0$$

(4) Use Rodrigues' Formula to derive the first three Legendre Polynomials $P_0(x)$, $P_1(x)$, and $P_2(x)$, and compare to the results given in the course notes.

(5) Two masses $3m$ and $2m$ are connected to two identical springs as shown:



The masses are free to move horizontally and one spring is attached to a fixed wall.

(a) Write down Newton's Second Law for each of the two masses.

(b) Find the eigenfrequencies and describe the motion of the two eigenmodes.

(c) Write $x_1(t)$ and $x_2(t)$ in terms of four arbitrary constants a, b, c, and d.

(d) Make a plot of $x_1(t)$ and $x_2(t)$ subject to the initial conditions $x_1(0) = 1$, and $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$. (You can let Mathematica solve for a, b, c, and d.)

(e) For the example in class, we found that the two combinations $x_\pm(t) = x_1(t) \pm x_2(t)$ oscillated with the two eigenfrequencies. What linear combinations $x_A(t)$ and $x_B(t)$ of $x_1(t)$ and $x_2(t)$ oscillate with the eigenfrequencies in this case? The answer should be clear from (c) above. Plot $x_A(t)$ and $x_B(t)$ and show they they oscillate with the correct frequencies.

(1) (a) The two solutions in this case are $y_1(x) = x^{r_1}$ and $y_2(x)x^{r_2}$ where r_1 and r_2 are distinct real numbers. The Wronskian is

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = r_2x^{r_1+r_2-1} - r_1x^{r_1+r_2-1} = (r_2 - r_1)x^{r_1+r_2-1}$$

which is nonzero for x away from the singular point at $x = 0$.

(b) For $(\alpha - 1)^2 = 4\beta^2$, there is one root $r = (1 - \alpha)/2 = \beta$ to the characteristic equation giving $y(x) = x^\beta$. (If we were to choose $-\beta$, then α would be different and it would not be the same differential equation.) To get a second solution, write $y(x) = u(x)x^\beta$ and so

$$\begin{aligned}
y'(x) &= u'(x)x^\beta + \beta u(x)x^{\beta-1} \\
\alpha xy'(x) = (1-2\beta)xy'(x) &= (1-2\beta)u'(x)x^{\beta+1} + \beta(1-2\beta)u(x)x^\beta \\
y''(x) &= u''(x)x^\beta + 2\beta u'(x)x^{\beta-1} + \beta(\beta-1)u(x)x^{\beta-2} \\
x^2y''(x) &= u''(x)x^{\beta+2} + 2\beta u'(x)x^{\beta+1} + \beta(\beta-1)u(x)x^\beta \\
x^2y''(x) + \alpha xy'(x) + \beta^2 y(x) &= u''(x)x^{\beta+2} + \left[2\beta + (1-2\beta)\right]u'(x)x^{\beta+1} \\
&\quad + \left[\beta(\beta-1) + \beta(1-2\beta) + \beta^2\right]u(x)x^\beta \\
&= \left[xu''(x) + u'(x)\right]x^{\beta+1} = 0
\end{aligned}$$

Therefore $xu''(x) + u'(x) = 0$, or $xv'(x) + v(x) = 0$ where $v(x) = u'(x)$. This separable first order equation for $v(x)$ is easily solved by

$$\frac{dv}{v} = -\frac{dx}{x} \qquad \text{so} \qquad \log v(x) = -\log x + \log c_2 = \log\frac{c_2}{x}$$

so that $v(x) = u'(x) = c_2/x$ and $u(x) = c_1 + c_2\log x$. This means that the two presumably independent solutions are $y_1(x) = x^\beta$ and $y_2(x) = x^\beta \log x$. The Wronskian is

$$\begin{aligned}
W(x) &= y_1(x)y_2'(x) - y_2(x)y_1'(x) \\
&= x^\beta\left[x^{\beta-1} + \beta x^{\beta-1}\log x\right] - x^\beta\log x\left[\beta x^{\beta-1}\right] = x^{2\beta-1}
\end{aligned}$$

which, again, is nonzero for x away from the singular point at $x = 0$

(2) First let's deal with $n!$ for $n < 0$. From $n! = \Gamma(n+1)$ and putting $m = -n > 0$,

$$n! = \int_0^\infty x^{n-1} e^{-x} dx = \int_0^\infty \frac{e^{-x}}{x^{m+1}} dx$$

which diverges for all positive m because the $x \to 0$ limit of the integrand becomes $1/x^{m+1}$. Therefore we take $n! \to \infty$ for $n < 0$, that is $1/n! = 0$. (I think this would actually show up naturally if we derived the Bessel Function expansion explicitly for integer order $m < 0$.) Now the series expansion for the Bessel Function is

$$J_m(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{m+2k}$$

For $p = -m > 0$, $(-p+k)!$ will be infinite until $k = p$. Therefore, with $n = k - p$,

$$J_m(x) = \sum_{k=p}^{\infty} \frac{(-1)^k}{k!(-p+k)!} \left(\frac{x}{2}\right)^{-p+2k} = \sum_{n=0}^{\infty} \frac{(-1)^{n+p}}{(n+p)!n!} \left(\frac{x}{2}\right)^{2k+p} = (-1)^p J_p(x) = (-1)^{-m} J_{-m}(x)$$

That is, $J_{-m}(x) = (-1)^m J_m(x)$. Therefore

$$y(x) = c_1 J_m(x) + c_2 J_{-m}(x) = y(x) = c_1 J_m(x) + c_2 (-1)^m J_m(x) = [c_1 + c_2(-1)^m]\, J_m(x)$$

so this represents only one solution, not two linearly independent ones.

(3) We have $R(r) = \sin(kr)/kr$ so

$$\begin{aligned}
R'(r) &= k\frac{\cos(kr)}{kr} - \frac{1}{k}\frac{\sin(kr)}{r^2} \\
R''(r) &= -k^2\frac{\sin(kr)}{kr} - \frac{\cos(kr)}{r^2} - \frac{\cos(kr)}{r^2} + \frac{2}{k}\frac{\sin(kr)}{r^3} \\
&= -k^2\frac{\sin(kr)}{kr} - 2\frac{\cos(kr)}{r^2} + \frac{2}{k}\frac{\sin(kr)}{r^3} \\
r^2 R''(r) + 2rR'(r) + k^2 r^2 R(r) &= -kr\sin(kr) - 2\cos(kr) + 2\frac{\sin(kr)}{kr} \\
&\quad +2\cos(kr) - 2\frac{\sin(kr)}{kr} + kr\sin(kr) = 0
\end{aligned}$$

(4) Rodrigues' Formula is (3.30), and is with the first few Legendre Polynomials on Page 67 of the notes as I write this. We have

$$
\begin{aligned}
P_0(x) &= \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1 \\
P_1(x) &= \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1)^1 = \frac{1}{2}2x = x \\
P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{8}(12x^2-4x) = \frac{1}{2}(3x^2-1)
\end{aligned}
$$

(5) The force on the first mass is $F_1 = -kx_1 + k(x_2 - x_1)$, exactly the same as the example in the notes. However, now the force on the second mass is $F_2 = -k(x_2 - x_1)$. Therefore we need to solve the differential equations

$$3x_1''(t) + 2\omega_0^2 x_1(t) - \omega_0^2 x_2(t) = 0 \qquad \text{and} \qquad 2x_2''(t) + \omega_0^2 x_2(t) - \omega_0^2 x_1(t) = 0$$

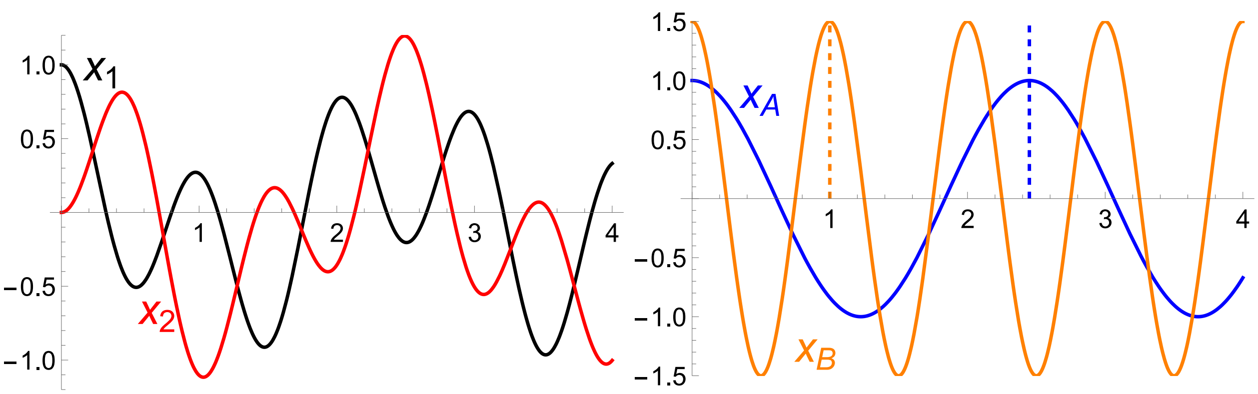
Inserting $x_1(t) = a_1 e^{i\omega t}$ and $x_2(t) = a_2 e^{i\omega t}$ results in

$$
\begin{aligned}
(2\omega_0^2 - 3\omega^2)a_1 - \omega_0^2 a_2 &= 0 \\
-\omega_0^2 a_1 + (\omega_0^2 - 2\omega^2)a_2 &= 0
\end{aligned}
$$

so $(2\omega_0^2 - 3\omega^2)(\omega_0^2 - 2\omega^2) - \omega_0^4 = \omega_0^4 - 7\omega_0^2\omega^2 + 6\omega^4 = (\omega_0^2 - \omega^2)(\omega_0^2 - 6\omega^2) = 0$. Therefore the eigenfrequencies are $\omega_1^2 = \omega_0^2$ and $\omega_2^2 = \omega_0^2/6$. The amplitude ratios are $a_2^{(1)}/a_1^{(1)} = -1$ (same magnitude and 180° out of phase) and $a_2^{(2)}/a_1^{(2)} = 3/2$ (in phase with a 3:2 ratio). So,

$$
\begin{aligned}
x_1(t) &= ae^{i\omega_0 t} + be^{-i\omega_0 t} + ce^{i\omega_0 t/\sqrt{6}} + de^{-i\omega_0 t/\sqrt{6}} \\
x_2(t) &= -ae^{i\omega_0 t} - be^{-i\omega_0 t} + \frac{3c}{2}e^{i\omega_0 t/\sqrt{6}} + \frac{3d}{2}e^{-i\omega_0 t/\sqrt{6}}
\end{aligned}
$$

I let Mathematica solve for a, b, c, and d for the given initial conditions. Plots of $x_1(t)$ and $x_2(t)$ are shown here:



From the expressions for $x_1(t)$ and $x_2(t)$ it is clear that the linear combinations are

$$
\begin{aligned}
x_A(t) &= x_1(t) + x_2(t) \qquad &\text{for} \qquad \omega = \omega_0/\sqrt{6} \\
\text{and} \qquad x_B(t) &= \frac{3}{2}x_1(t) - x_2(t) \qquad &\text{for} \qquad \omega = \omega_0
\end{aligned}
$$

Plots of $x_A(t)$ and $x_B(t)$ are above. The vertical lines mark periods of unity and $\sqrt{6}$.

PHYS2502 Mathematical Physics Homework #7 Due 8 Mar 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Derive the unit vectors $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ in spherical polar coordinates in terms of the Cartesian unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the spherical polar coordinates r, θ, and ϕ. Your starting point should be the transformation equations that give you the Cartesian coordinates x, y, and z in terms of the spherical coordinates.

(2) The kinetic energy of a particle of mass m is $K = mv^2/2 = m\vec{v}\cdot\vec{v}/2$, where $\vec{v} = d\vec{r}/dt$ is the particle's velocity vector. Derive an expression for K in terms of the spherical coordinates r, θ, and ϕ and their rates of change $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ with respect to time. Simplify your result as much as possible. You can carry all this out with the chain rule and the same transformation equations you used above in (1), but there is also a much simpler way.

(3) Find an expression for the square of the magnitude of the cross product $|\vec{A}\times\vec{B}|^2$ in terms of the magnitudes of $\vec{A}$ and $\vec{B}$ and their dot product $\vec{A}\cdot\vec{B}$, in two different ways:

(a) Directly from the definitions of the magnitudes of $|\vec{A}\times\vec{B}|$ and $\vec{A}\cdot\vec{B}$.

(b) Using components and the summation convention, along with the relationship between the totally antisymmetric symbol and the Kronecker delta.

(4) Show that the gradient operator in spherical coordinates is given by

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

You can do this using the transformation equations and the chain rule for partial derivatives, but you don't need to do nearly that much work. Think of the gradient as a "directional derivative" as discussed in the notes and in class, and use the results of Problem (1) above to write down what is the change $d\vec{r}$ for each of the three orthogonal directions in spherical coordinates.

(5) Derive the Laplacian in plane polar coordinates, i.e. cylindrical coordinates with no z-component. That is, show that

$$\vec{\nabla}^2 f(r,\phi) = \vec{\nabla}\cdot\vec{\nabla}f(r,\phi) = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{r}\frac{\partial f}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial f}{\partial\phi}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial f}{\partial\phi^2}$$

Don't forget that you need to take into account that $\hat{r}$ and $\hat{\phi}$ depend explicitly on ϕ. Writing these unit vectors in terms of $\hat{\imath}$ and $\hat{\jmath}$ is probably the easiest way to do this.

(1) The spherical coordinate transformation equations are

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

Therefore, infinitesimal changes in the Cartesian coordinates are

$$\begin{aligned} dx &= dr\,\sin\theta\cos\phi + rd\theta\,\cos\theta\cos\phi - rd\phi\,\sin\theta\sin\phi \\ dy &= dr\,\sin\theta\sin\phi + rd\theta\,\cos\theta\sin\phi + rd\phi\,\sin\theta\cos\phi \\ dz &= dr\,\cos\theta - rd\theta\,\sin\theta \end{aligned}$$

We can now write down $d\vec{r}$ and group the terms as follows

$$\begin{aligned} d\vec{r} &= \hat{\imath}\,dx + \hat{\jmath}\,dy + \hat{k}\,dz \\ &= \hat{\imath}\,[dr\,\sin\theta\cos\phi + rd\theta\,\cos\theta\cos\phi - rd\phi\,\sin\theta\sin\phi] \\ &+ \hat{\jmath}\,[dr\,\sin\theta\sin\phi + rd\theta\,\cos\theta\sin\phi + rd\phi\,\sin\theta\cos\phi] \\ &+ \hat{k}\,[dr\,\cos\theta - rd\theta\,\sin\theta] \\ &= dr\,[\hat{\imath}\,\sin\theta\cos\phi + \hat{\jmath}\,r\sin\theta\sin\phi + \hat{k}\,\cos\theta] \\ &+ rd\theta\,[\hat{\imath}\,\cos\theta\cos\phi + \hat{\jmath}\,\cos\theta\sin\phi - \hat{k}\,\sin\theta] \\ &+ r\sin\theta\,d\phi\,[-\hat{\imath}\,\sin\phi + \hat{\jmath}\,\cos\phi] \end{aligned}$$

where we recognize that dr is the change in magnitude of $d\vec{r}$ when θ and ϕ are held constant; $rd\theta$ when r and ϕ are held constant; and $r\sin\theta\,d\phi$ when r and θ are held constant. (You can also just force the component to give you a unit magnitude for $\hat{\phi}$.) Therefore

$$\begin{aligned} \hat{r} &= \hat{\imath}\,\sin\theta\cos\phi + \hat{\jmath}\,r\sin\theta\sin\phi + \hat{k}\,\cos\theta \\ \hat{\theta} &= \hat{\imath}\,\cos\theta\cos\phi + \hat{\jmath}\,\cos\theta\sin\phi - \hat{k}\,\sin\theta \\ \hat{\phi} &= -\hat{\imath}\,\sin\phi + \hat{\jmath}\,\cos\phi \end{aligned}$$

and we write

$$d\vec{r} = \hat{r}\,dr + \hat{\theta}\,rd\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

(2) We need to find the expression for $\vec{v} = d\vec{r}/dt = \dot{\vec{r}}$ in spherical coordinates, and then take the dot product of it with itself. So, we need the derivatives

$$\begin{aligned}
\dot{x} &= \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \\
\dot{y} &= \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \\
\dot{z} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta
\end{aligned}$$

It's tempting to just peck this into MATHEMATICA to simply the squares, but...

$$\begin{aligned}
\dot{x}^2 &= \dot{r}^2\sin^2\theta\cos^2\phi + r^2\dot{\theta}^2\cos^2\theta\cos^2\phi + r^2\dot{\phi}^2\sin^2\theta\sin^2\phi \\
&+ 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\phi - 2r\dot{r}\dot{\phi}\sin^2\theta\cos\phi\sin\phi - 2r^2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi \\
\dot{y}^2 &= \dot{r}^2\sin^2\theta\sin^2\phi + r^2\dot{\theta}^2\cos^2\theta\sin^2\phi + r^2\dot{\phi}^2\sin^2\theta\cos^2\phi \\
&+ 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\sin^2\phi + 2r\dot{r}\dot{\phi}\sin^2\theta\cos\phi\sin\phi + 2r^2\dot{\theta}\dot{\phi}\cos\theta\sin\theta\cos\phi\sin\phi \\
\dot{x}^2 + \dot{y}^2 &= \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\phi}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta \\
\dot{z}^2 &= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta
\end{aligned}$$

Therefore, the kinetic energy is

$$K = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$$

The simple way to do this problem is to just use the results of (1) above. We saw that

$$d\vec{r} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta d\phi\,\hat{\phi} \qquad \text{so} \qquad \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\dot{\phi}\,\hat{\phi}$$

which immediately gives $\dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$.

(3) (a) For two vectors $\vec{A}$ and $\vec{B}$ with a planar angle ψ between them, $|\vec{A}\times\vec{B}| = AB\sin\psi$ and $\vec{A}\cdot\vec{B} = AB\cos\psi$, where I simply write $A = |\vec{A}|$ and $B = |\vec{B}|$. Therefore

$$|\vec{A}\times\vec{B}|^2 = A^2B^2\sin^2\psi = A^2B^2(1-\cos^2\psi) = A^2B^2 - (\vec{A}\cdot\vec{B})^2$$

(b) In coordinates and the summation convention $(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$ and $\vec{A}\cdot\vec{B} = A_iB_i$, so

$$\begin{aligned}
|\vec{A}\times\vec{B}|^2 &= \epsilon_{ijk}A_jB_k\epsilon_{imn}A_mB_n = \epsilon_{ijk}k\epsilon_{imn}A_jB_kA_mB_n \\
&= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})A_jB_kA_mB_n = A_jB_kA_jB_k - A_jB_kA_kB_j \\
&= (A_jA_j)(B_kB_k) - (A_jB_j)(A_kB_k) = A^2B^2 - (\vec{A}\cdot\vec{B})^2
\end{aligned}$$

(4) Reiterating from Problem (1), we have

$$d\vec{r} = \hat{r}\, dr + \hat{\theta}\, r d\theta + \hat{\phi}\, r \sin\theta\, d\phi$$

That is, the change in the r-direction is dr, the change in the θ-direction is $rd\theta$, and the change in the ϕ-direction is $r\sin\theta\, d\phi$. Therefore, the gradient operator is

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

If you want to do this problem by taking all of the partial derivative chain rule, that's certainly possible, but very tedious. Maybe it's a good exercise to work it through in MATHEMATICA.

(5) The notes derive the unit vectors in plane polar coordinates as

$$\hat{r} = \hat{\imath}\cos\phi + \hat{\jmath}\sin\phi \qquad \text{and} \qquad \hat{\phi} = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi$$

Notice that these unit vectors depend on ϕ but not on r. In fact, $\partial\hat{r}/\partial\phi = \hat{\phi}$ and $\partial\hat{\phi}/\partial\phi = -\hat{r}$. Now put these into the expression we're given and plow through to get

$$\begin{aligned}
\vec{\nabla}^2 f(r,\phi) &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{r}\frac{\partial f}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial f}{\partial\phi}\right) \\
&= \hat{r}\frac{\partial}{\partial r}\cdot\left(\hat{r}\frac{\partial f}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial f}{\partial\phi}\right) + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial\phi}\cdot\left(\hat{r}\frac{\partial f}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial f}{\partial\phi}\right) \\
&= \hat{r}\cdot\left(\hat{r}\frac{\partial^2 f}{\partial r^2} - \hat{\phi}\frac{1}{r^2}\frac{\partial f}{\partial\phi} + \hat{\phi}\frac{1}{r}\frac{\partial^2 f}{\partial r\partial\phi}\right) \\
&\quad + \hat{\phi}\frac{1}{r}\cdot\left(\frac{\partial\hat{r}}{\partial r}\frac{\partial f}{\partial r} + \hat{r}\frac{\partial^2 f}{\partial r^2} + \frac{\partial\hat{\phi}}{\partial\phi}\frac{1}{r}\frac{\partial f}{\partial\phi} + \hat{\phi}\frac{1}{r}\frac{\partial^2 f}{\partial\phi^2}\right) \\
&= \frac{\partial^2 f}{\partial r^2} - 0 + 0 \\
&\quad + \hat{\phi}\frac{1}{r}\cdot\left(\hat{\phi}\frac{\partial f}{\partial r} + \hat{r}\frac{\partial^2 f}{\partial r^2} - \hat{r}\frac{1}{r}\frac{\partial f}{\partial\phi} + \hat{\phi}\frac{1}{r}\frac{\partial^2 f}{\partial\phi^2}\right) \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + 0 - 0 + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2}
\end{aligned}$$

PHYS2502 Mathematical Physics Homework #8 Due 15 Mar 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) For a scalar field $f(\vec{r})$ over some volume V in three dimensional space, prove that

$$\int_V \vec{\nabla} f \, dV = \oint_S f \, d\vec{S}$$

where S is the surface enclosing V. You can use the "cut the volume up into little bricks" approach we used in class, or you can try inventing a vector field $\vec{A} = f\vec{C}$ where $\vec{C}$ is some arbitrary constant vector, and then use a different surface theorem.

(2) Calculate the curl of the following vector field in both Cartesian coordinates and cylindrical polar coordinates:

$$\vec{A}(\vec{r}) = -\frac{\hat{i}y - \hat{j}x}{x^2 + y^2} = \frac{\hat{\phi}}{r}$$

Now calculate directly the line integral $\oint \vec{A} \cdot d\vec{\ell}$ around a closed circle of radius R in the xy plane, centered at the origin. (I suggest you do this with the polar coordinate expression.) Can you reconcile these two seemingly inconsistent results?

(3) Calculate the divergence of the following vector field in both Cartesian coordinates and spherical polar coordinates:

$$\vec{A}(\vec{r}) = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\hat{r}}{r^2}$$

Now calculate directly the surface integral $\oint \vec{A} \cdot d\vec{S}$ around a sphere of radius R, centered at the origin. (I suggest you do this with the polar coordinate expression.) Can you reconcile these two seemingly inconsistent results?

(4) Use the "Separation of Variables" approach to solve the partial differential equation

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

for the function $u(x,t)$ with the initial condition $u(x,0) = \sin(\pi x/2)$ and boundary conditions $u(2,t) = u(0,t) = 0$.

(5) Look for a solution to the Helmholtz Equation $\vec{\nabla}^2 f(r,\phi) + k^2 f(r,\phi) = 0$ in plane polar coordinates by writing $f(r,\phi) = R(r)\Phi(\phi)$. Now insist that $\Phi(\phi + 2\pi) = \Phi(\phi)$, that is $\Phi(\phi)$ must be "single valued", and show that solutions for $R(r)$ must be Bessel Functions $J_m(kr)$ of integer order m.

PHY2502 Mathematical Physics Homework #8 Solutions

(1) The "little bricks" derivation is simple. Consider first just the x-component of a little brick. The two contributions are $f(x+dx,y,z)\,dy\,dz\,\hat{\imath}$ on one side, and $f(x,y,z)\,dy\,dz\,(-\hat{\imath})$ on the other side, so the x-component of the surface integral is

$$f(x+dx,y,z)\,dy\,dz-(x,y,z)\,dy\,dz=\frac{f(x+dx,y,z)-(x,y,z)}{dx}dx\,dy\,dz\to\frac{\partial f}{\partial x}dV$$

The surface integrals cancel on adjacent bricks, leaving only the outside surface when we add things up into an integral. Similarly for the y- and z- components, and we're done.

The other way is slicker. Use the divergence theorem. Since $\vec{\nabla}\cdot\vec{A}=\vec{C}\cdot\vec{\nabla}f$ (which is simple to prove just by looking at the components), we have

$$\int_V\vec{\nabla}\cdot\vec{A}\,dV=\vec{C}\cdot\int_V\vec{\nabla}f\,dV=\oint_S\vec{A}\cdot d\vec{S}=\vec{C}\cdot\oint_S f\,d\vec{S}$$

From here, you can write that

$$\vec{C}\cdot\left[\int_V\vec{\nabla}f\,dV-\oint_S f\,d\vec{S}\right]=0$$

and argue that since $\vec{C}$ is arbitrary, then the expression in square brackets must be zero. Or you could go through this equation component by component, setting $\vec{C}=\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

(2) In Cartesian coordinates, there is only a z-component to the curl since $\vec{A}$ has no z-component and there is no z-dependence to it. We have

$$\begin{aligned}(\vec{\nabla}\times\vec{A})_z &= \frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=\frac{1}{x^2+y^2}-\frac{x}{(x^2+y^2)^2}(2x)+\frac{1}{x^2+y^2}-\frac{y}{(x^2+y^2)^2}(2y)\\ &= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}=0\end{aligned}$$

In cylindrical coordinates, there is only a ϕ-component and it only depends on r. It is clear from the formula for the curl that there is, again, only a z-component, and

$$(\vec{\nabla}\times\vec{A})_z=\frac{1}{r}\frac{\partial}{\partial r}(rA_phi)=\frac{1}{r}\frac{\partial}{\partial r}(1)=0$$

OK, the curl is zero. However, for the circle, $d\vec{\ell}=\hat{\phi}\,Rd\phi$ so

$$\oint\vec{A}\cdot d\vec{\ell}=\oint\frac{1}{R}R\,d\phi=2\pi$$

which is most certainly not zero. It seems that Stokes' Theorem implies that $0=2\pi$.

Such nonsense is avoided because the curl is zero everywhere *except* at $x=y=0$. It is clearly undefined there, because the denominator of the field goes to zero. The solution turns out to be defined in terms of the Dirac δ-function, which we will cover later. Indeed, this is the problem of a magnetic field from an infinitely long straight wire.

(3) In Cartesian coordinates, the divergence is

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{A} &= \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} \\
&= \frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3}{2}\frac{x}{(x^2+y^2+z^2)^{5/2}}2x \\
&+ \frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3}{2}\frac{y}{(x^2+y^2+z^2)^{5/2}}2y \\
&+ \frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3}{2}\frac{z}{(x^2+y^2+z^2)^{5/2}}2z \\
&= \frac{3(x^2+y^2+z^2)-3x^2-3y^2-3z^2}{(x^2+y^2+z^2)^{5/2}}=0
\end{aligned}
$$

In spherical coordinates, there is only the r-component so

$$\vec{\nabla}\cdot\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)=\frac{1}{r^2}\frac{\partial}{\partial r}(1)=0$$

So the divergence is zero. However, if we consider the volume enclosed by a sphere of radius R centered at the origin, then the surface element is $d\vec{S}=\hat{r}\,dS$ and

$$\oint_S \vec{A}\cdot d\vec{S}=\frac{1}{R^2}\oint_S dS=\frac{1}{R^2}4\pi R^2=4\pi$$

which is not zero. Again, it looks like Gauss' Theorem is violated. However, we cannot say the divergence is zero at the origin where the field tends to infinity. Indeed, it's another Dirac δ-function. This is the field of a point electric charge, of course.

(4) Start by writing $u(x,t)=X(x)T(t)$ which leads to

$$4\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{T}\frac{dT}{dt}$$

Using the standard argument, the left side only depends on x and the right only depends on t, so both sides must equal a constant. If you call the constant $-k^2$ where k is real, then

$$\frac{d^2X}{dx^2}=-\frac{k^2}{4}X \qquad \text{so} \qquad X(x)=a\cos\left(\frac{kx}{2}\right)+b\sin\left(\frac{kx}{2}\right)$$

We also have

$$\frac{dT}{dt}=-k^2T \qquad \text{so} \qquad T(t)=ce^{-k^2t}$$

Redefining a and b to absorb c, we have the general solution

$$u(x,t)=\left[a\cos\left(\frac{kx}{2}\right)+b\sin\left(\frac{kx}{2}\right)\right]e^{-k^2t}$$

Clearly we need $k=\pi$ to satisfy the initial condition $u(x,0)=\sin(\pi x/2)$. In this case

$$u(x,t)=\left[a\cos\left(\frac{\pi x}{2}\right)+b\sin\left(\frac{\pi x}{2}\right)\right]e^{-\pi^2t}$$

Setting $u(0,t)=0$ implies that $a=0$, so $b=1$ for the boundary condition and the final solution is

$$u(x,t)=\sin\left(\frac{\pi x}{2}\right)e^{-\pi^2t}$$

(5) The Helmholtz Equation in plane polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial f}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2}=-k^2 f(r,\phi)$$

Substituting $f(r,\phi)=R(r)\Phi(\phi)$, and multiplying through by $r^2/R\Phi$, you arrive at

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left[r\frac{dR}{dr}\right]+\frac{1}{\Phi}\frac{1}{r^2}\frac{d^2\Phi}{d\phi^2}=-k^2R\Phi \qquad \text{so} \qquad \frac{1}{R}r\frac{d}{dr}\left[r\frac{dR}{dr}\right]+k^2r^2=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

and we have the familiar situation where both sides of the equation have to equal some constant in order to be equal to each other. If you set the constant equal to m^2, then

$$\frac{d^2\Phi}{d\phi^2}=-m^2\Phi(\phi) \qquad \text{so} \qquad \Phi(\phi)=e^{im\phi}$$

where we ignore a possible constant factor since we aren't concerned with boundary conditions. We also recognize that m can be either positive or negative, and won't bother with the two separate solutions. Now to enforce single valuedness, $\Phi(\phi+2\pi)=\Phi(\phi)$, we need $e^{im2\pi}=\cos(m2\pi)+i\sin(m2\pi)=1$ so m needs to be an integer. The radial equation becomes

$$r\frac{d}{dr}\left[r\frac{dR}{dr}\right]+k^2r^2=r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+k^2r^2=m^2R(r)$$

If we write $x\equiv kr$ and define $Q(x)=R(r)$ then $r(d/dr)=x(d/dx)$, $r^2(d^2/dr^2)=x^2(d/dx^2)$ and this equation becomes

$$x^2\frac{d^2Q}{dx^2}+x\frac{dQ}{dx}+(x^2-m^2)Q=0$$

which in fact is Bessel's Equation of order m, where m is an integer.

PHYS2502 Mathematical Physics Homework #9 Due 22 Mar 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A function $u(x,t)$ satisfies the wave equation in one dimension x with velocity v. The initial conditions are $u(x,0) = p(x)$, for an arbitrary function $p(x)$, and $\dot{u}(x,0) = 0$. Show that the time development of the wave corresponds to the "splitting" of $p(x)$ into two pieces, one moving to the left and the other moving to the right, each being an exact copy of $p(x)$ but divided by two.

(2) A wave $u(x,t) = g(x+vt)$ moves to the left along a string on the positive x-axis.

(a) Assume the string is fixed at $x = 0$ so that it cannot move, that is $u(0,t) = 0$. Find the motion of the string for $x \geq 0$ for all times.

(b) Now ssume the string is free to move up and down at $x = 0$, and does so in a way that it is always horizontal, that is $\partial u(x,t)/\partial x|_{x=0} = 0$. Once again find the motion of the string for $x \geq 0$ for all times.

(3) Use a Fourier Sine decomposition to find the motion of a string that is fixed at $x = 0$ and $x = L$, and that starts from rest with an initial shape $u(x,0) = (2/L)^4 x^2 (x-L)^2$. (You'll want to use MATHEMATICA for this problem.) Follow the procedure in the notes for the lopsided triangle wave, and compare to your numerical solution from Lab #8. I encourage you do do this with an animation.

(4) This problem concerns the Fourier Transform and width of the Gaussian function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

(a) Find the Fourier Transform $A(k)$ of $f(x)$. The integral is not hard to do. Just complete the square in the exponent, and use what you know about Gaussian integrals.

(b) Calculate the "width" Δx (that is, the square root of the variance) of $f(x)$. (See the notes for details.) Again make use of what you know about Gaussian integrals.

(c) Next find the width Δk of $A(k)$.

(d) Determine the product $\Delta x \Delta k$. How does this compare to the same result for the triangle pulse that we derived in class?

(5) Show that the following relationships are consistent with the fundamental definition of the δ-function. You can make use of results derived in the notes, but you'll likely find it useful to emply integration by parts.

(a) $x\delta(x) = 0$

(b) $x\delta'(x) = -\delta(x)$

(c) $x^2\delta''(x) = 2\delta(x)$

PHY2502 Mathematical Physics Homework #9 Solutions

(1) Start with the general solution $u(x,t) = f(x - vt) + g(x + vt)$ and follow your nose.

$$\begin{aligned}
u(x,t) &= f(x - vt) + g(x + vt) \\
u(x,0) &= f(x) + g(x) = p(x) \\
\dot{u}(x,t) &= -vf'(x - vt) + vg'(x + vt) \\
\dot{u}(x,0) &= -vf'(x) + vg'(x) = 0
\end{aligned}$$

For the last equation, divide by v and integrate to get $-f(x) + g(x) = c$. Add this to the second equation above $f(x) + g(x) = p(x)$ to find $2g(x) = p(x) + c$ or $g(x) = p(x)/2 + c/2$. This leads to $f(x) = p(x) - g(x) = p(x)/2 - c/2$, so the final solution is

$$u(x,t) = \frac{1}{2}\left[p(x - vt) - c\right] + \frac{1}{2}\left[p(x + vt) + c\right] = \frac{1}{2}p(x - vt) + \frac{1}{2}p(x + vt)$$

which is, mathematically, precisely what we were asked to show.

(2) (a) The solution is $u(x,t) = g(x + vt) - g(-x + vt)$ which is clearly a solution to the wave equation and which satisfies $u(0,t) = 0$. Physically, we combine a rightward moving wave in the "virtual" space $x \leq 0$ with the leftward moving wave in the physical space. The wave reflects at $x = 0$ and reverses the sign of the shape function $g(z)$. (b) This time the solution is $u(x,t) = g(x + vt) + g(-x + vt)$ since $\partial u/\partial x = g'(x + vt) - g'(-x + vt)$ which is zero at $x = 0$. The wave again reflects at $x = 0$, but the sign of the shape function is unchanged.

(3) See the Mathematica notebook. The result looks just like the numerical solution from the lab exercise.

(4) Recall two results from Section 1.5.6 of the notes, namely

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \qquad \text{and} \qquad \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

You can find the Fourier Transform exactly as suggested in the assignment, namely

$$A(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2 - ikx} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-ik\sigma^2)^2/2\sigma^2} e^{-k^2\sigma^2/2} dx$$

The integral can be done trivially with the substitution $y = x - ik\sigma$ which turns it into the standard Gaussian integral, so

$$A(k) = \frac{1}{\sigma\sqrt{2\pi}} \sigma\sqrt{2\pi} e^{k^2\sigma^2/2} = e^{-\sigma^2 k^2/2}$$

which is just another Gaussian. The width of $f(x)$ is also easy to calculate. Since

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\, dx = \frac{1}{\sigma\sqrt{2\pi}} \sqrt{\pi 2\sigma^2} = 1$$

we just go ahead with the integral of x^2 and write

$$(\Delta x)^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2} = \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{2} \sqrt{\pi 8\sigma^6} = \sigma^2$$

In other words $\Delta x = \sigma$. Proceeding similarly to get Δk, first we calculate

$$\int_{-\infty}^{\infty} A(k)\, dk = \int_{-\infty}^{\infty} e^{-\sigma^2 k^2/2}\, dk = \sqrt{\frac{2\pi}{\sigma^2}} = \frac{\sqrt{2\pi}}{\sigma}$$

Now we can calculate the width of the Fourier Transform as

$$(\Delta k)^2 = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^2 A(k)\, dk = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^2 e^{-\sigma^2 k^2/2}\, dk = \frac{\sigma}{\sqrt{2\pi}} \frac{1}{2} \sqrt{\frac{8\pi}{\sigma^6}} = \frac{1}{\sigma^2}$$

which is a result we might have guessed. Therefore $\Delta k = 1/\sigma$ and $\Delta x \Delta k = 1$. This is smaller than for the triangle wave. In fact, the Gaussian pulse gives the smallest possible product. Essentially, this is the uncertainty principle from quantum mechanics.

(5) Firstly, for (a), we just use

$$\int_{-\epsilon}^{\epsilon} f(x)\delta(x)\,dx = f(0) \qquad \text{so} \qquad \int_{-\epsilon}^{\epsilon} x\delta(x)\,dx = x|_{x=0} = 0$$

By definition, the function $\delta(x)$ is zero for all $x \neq 0$, but is "large enough" at $x = 0$ so that its definite integral over a range including $x = 0$ gives unity. Therefore, for (b),

$$\int_{-\epsilon}^{\epsilon} x\delta'(x)\,dx = [x\delta(x)]_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon} (1)\delta(x)\,dx = -\int_{-\epsilon}^{\epsilon} \delta(x)\,dx$$

so $x\delta'(x) = -\delta(x)$. For (c), make use of (b) and get

$$\int_{-\epsilon}^{\epsilon} x^2\delta''(x)\,dx = \left[x^2\delta'(x)\right]_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon} (2x)\delta'(x)\,dx = [-x\delta(x)]_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} 2\delta(x)\,dx = \int_{-\epsilon}^{\epsilon} 2\delta(x)\,dx$$

so $x^2\delta''(x) = 2\delta(x)$

PHYS2502 Mathematical Physics Homework #10 Due 29 Mar 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Find the inverse $\underline{\underline{A}}^{-1}$ for the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

by solving the system of equations $\underline{\underline{A}}\,\underline{x} = \underline{c}$ for the vector $\underline{x}$ in terms of an arbitrary vector $\underline{c}$ and expressing your result as $\underline{x} = \underline{\underline{A}}^{-1}\underline{c}$. Try using MatrixForm[Inverse[A]] in MATHEMATICA to check your answer.

(2) Use the orthogonality of the Legendre Polynomials to derive an infinite series that gives $f(x) = e^x$ over the domain $-1 \leq x \leq 1$ in the form of a "Shin expansion"

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$

Plot your result for some number of terms of the series and compare to $f(x)$. Note that it is natural to carry out this calculation using MATHEMATICA, but you might be surprised at how few terms you need to get a good approximation.

(3) The trace $\mathsf{tr}(\underline{\underline{A}})$ of a matrix $\underline{\underline{A}}$ is defined as the sum over the diagonal elements of $\underline{\underline{A}}$, that is $\mathsf{tr}(\underline{\underline{A}}) = A_{ii}$. Prove that $\mathsf{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathsf{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$ for any two matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$, regardless of whether or not they commute.

(4) Consider rotations in 3D space, recalling how we describe rotations in a plane.

(a) Find the 3×3 matrix $\underline{\underline{A}}$ that rotates a vector by 90° around the z-axis.

(b) Find the 3×3 matrix $\underline{\underline{B}}$ that rotates a vector by 90° around the x-axis.

(c) Show that $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ by explicit matrix multiplication.

(d) For the vector $\underline{v} = \hat{\jmath}$, the unit vector in the y-direction, calculate $\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{v}$ and $\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{v}$. Sketch diagrams that demonstrate these unequal results. (Don't be too concerned about the sign of the rotation angle.)

(5) Find the norms of the vectors $\underline{v}$ and $\underline{u}$ below, and also show that they are orthogonal to each other. Then find some vector $\underline{w}$ with unit norm that is orthogonal to both $\underline{v}$ and $\underline{u}$.

$$\underline{v} = \begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} 2i \\ -2 \\ 0 \end{bmatrix}$$

PHY2502 Mathematical Physics Homework #10 Solutions

(1) The system of linear equations we need to solve is

$$\begin{aligned} x+y+z &= a \\ x+z &= b \\ x-z &= c \end{aligned}$$

The last two give $x=(b+c)/2$ and $z=(b-c)/2$, so

$$y = a-x-z = a-\frac{b+c}{2}-\frac{b-c}{2} = a-b$$

so the solution is

$$\begin{aligned} x &= \frac{1}{2}b+\frac{1}{2}c \\ y &= a-b \\ z &= \frac{1}{2}b-\frac{1}{2}c \end{aligned}$$

which says that

$$\underline{\underline{A}}^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

which agrees with the answer from Mathematica.

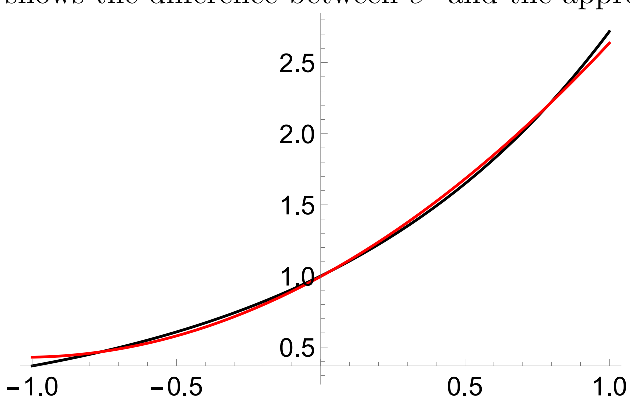
(2) The orthogonality relationship for the Legendre polynomials is

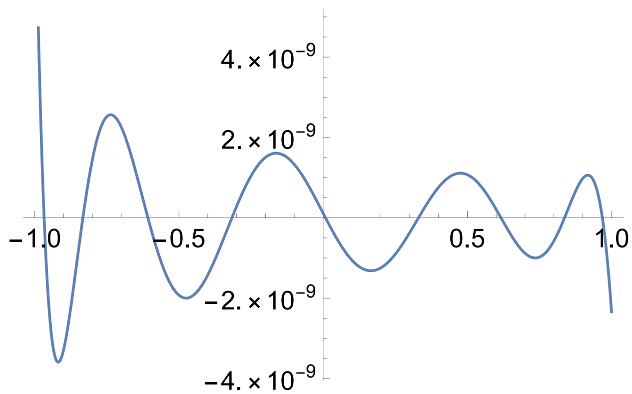
$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm}$$

This shows right away that the coefficients in the Shin expansion are

$$A_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} f(x)P_\ell(x)\,dx$$

See the Mathematica notebook for the code used to make the plots below. On the left is the comparison between $f(x)=e^x$ and the result for $\ell_{\max}=2$. For $\ell_{\max}=10$, the right shows the difference between e^x and the approximation.





(3) $\mathsf{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = (\underline{\underline{A}}\,\underline{\underline{B}})_{ii} = A_{ij}B_{ji} = B_{ji}A_{ij} = (\underline{\underline{B}}\,\underline{\underline{A}})_{jj} = \mathsf{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$

(4) A rotation about the z-axis is a rotation in the xy plane, and we know how to do that:

$$\underline{\underline{A}} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\phi=90^\circ} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This does what you expect to the x, y, z unit vectors:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

That is, it turns a unit vector in the x-direction into the y-direction, y into $-x$, but leaves z unchanged. Similarly, a 90° rotation about the x-axis turns z into y and y into $-z$, so

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

So now multiply these two matrices in different order and see what happens.

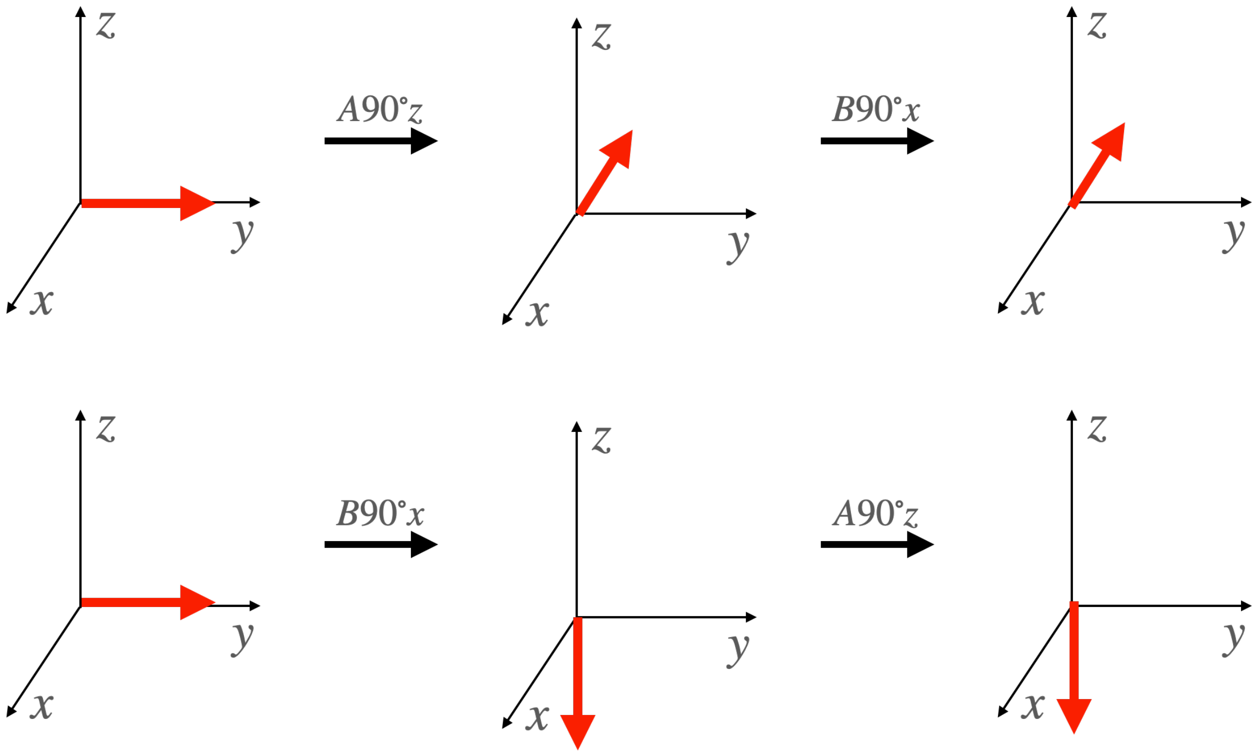
$$\begin{aligned} \underline{\underline{A}}\,\underline{\underline{B}} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \\ \underline{\underline{B}}\,\underline{\underline{A}} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

Indeed, these two matrices are not equal. Look at what these do to a y unit vector:

$$\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{e}_y = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{e}_y = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

That is $\underline{\underline{B}}\,\underline{\underline{A}}$ turns $\underline{e}_y$ into $-\underline{e}_x$, and $\underline{\underline{A}}\,\underline{\underline{B}}$ turns $\underline{e}_y$ into $-\underline{e}_z$.

The following diagrams show what this looks like physically. There is a little bit of trickiness to the sign of the angle, but that's not important.



(5) In terms of components, $\langle u|v\rangle = u_i^* v_i$, in general, so

$$\begin{aligned}
\langle v|v\rangle^{1/2} &= \sqrt{(-i)i + 1(1) + (-1)(-1)} = \sqrt{3} \\
\langle u|u\rangle^{1/2} &= \sqrt{(-2i)2i + (-2)(-2) + (0)(0)} = 2\sqrt{2} \\
\langle u|v\rangle &= (-2i)i - 2(1) + 0(-1) = 2 - 2 = 0
\end{aligned}$$

To find $\underline{w}$, we have the three equations

$$\begin{aligned}
i\,w_1 + w_2 - w_3 &= 0 \\
2i\,w_1 - 2\,w_2 &= 0 \\
|w_1|^2 + |w_2|^2 + |w_3|^2 &= 1
\end{aligned}$$

so $w_2 = iw_1$ and $w_3 = iw_1 + w_2 = 2iw_1$. Therefore

$$|w_1|^2 + |w_2|^2 + |w_3|^2 = |w_1|^2 + |w_1|^2 + |2w_1|^2 = 6|w_1|^2 = 1$$

so any w_1 with modulus $1/\sqrt{6}$ will do. We might as well make it real, so

$$\underline{w} = \begin{bmatrix} 1/\sqrt{6} \\ i/\sqrt{6} \\ 2i/\sqrt{6} \end{bmatrix}$$

PHYS2502 Mathematical Physics Homework #11 Due 5 Apr 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Use the properties of determinants to prove that $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{0}}$ implies that either $|\underline{\underline{A}}| = 0$ or $|\underline{\underline{B}}| = 0$, or both, where $\underline{\underline{0}}$ is the matrix of all zeros. Demonstrate this with the matrices

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad \underline{\underline{B}} = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}$$

where a and b can be any complex numbers. Which matrix has zero determinant?

(2) Follow the procedure we went through in class to find the symmetry axes of the conic section $6x^2 + 12xy + y^2 = 16$, and find the angle they make with the x, y axes. What kind of curve is this? A plot would be helpful. You can do this with MATHEMATICA if you want, but the necessary algebra is rather simple.

(3) Find the eigenvalues and eigenvectors for the matrix

$$\underline{\underline{\sigma}}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

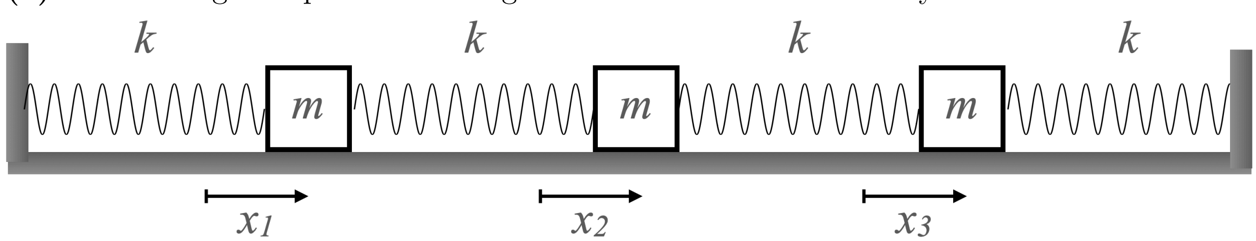
one of the three *Pauli matrices*. Do this by hand, not with MATHEMATICA. Normalize the eigenvectors and show that they are orthogonal.

(4) Find the eigenvalues, two of which equal each other, of the real symmetric matrix

$$\underline{\underline{A}} = \begin{bmatrix} 13 & 4 & -2 \\ 4 & 13 & -2 \\ -2 & -2 & 10 \end{bmatrix}$$

Construct the three eigenvectors by hand, not with MATHEMATICA. You will find you have more freedom than you would have thought. Do you see how to use this freedom to make all three eigenvectors mutually orthogonal?

(5) Find the eigenfrequencies and eigenmodes for the mechanical system



Make a plot that shows the motions of each of the three masses, for the three sets of initial conditions where the masses start at rest with position given by each of the three eigenvectors. Briefly describe the motions of the three masses, for each of the eigenmodes.

(1) Since $|\underline{\underline{A}}\,\underline{\underline{B}}| = |\underline{\underline{0}}| = 0$ and $|\underline{\underline{A}}\,\underline{\underline{B}}| = |\underline{\underline{A}}||\underline{\underline{B}}|$, we must have $|\underline{\underline{A}}||\underline{\underline{B}}| = 0$, so either $|\underline{\underline{A}}| = 0$ or $|\underline{\underline{B}}| = 0$, or both. For the two given matrices

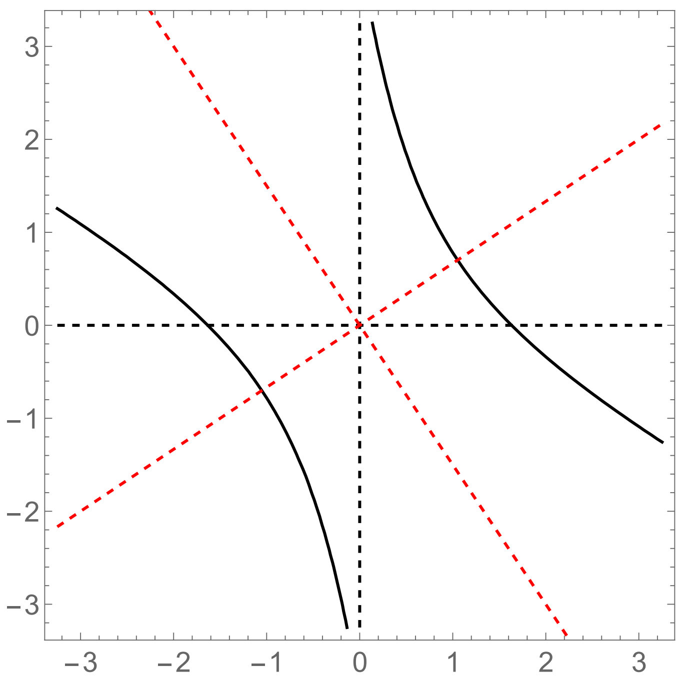
$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ -a & -b \end{bmatrix} = \begin{bmatrix} a-a & b-b \\ 2a-2a & 2b-2b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The second row of $\underline{\underline{A}}$ is a constant (2) times the first row, and the second row of $\underline{\underline{B}}$ is a constant (-1) times the first row, so the determinants of both matrices are zero.

(2) It is a simple matter to modify the "Tilted Ellipse" example from class (which is posted on the website) to run this problem, so that's what I did. See the accompanying MATHEMATICA notebook. The eigenvalues are 10 and -3, and opposite signs mean the curve is a hyperbola. The rotation matrix is

$$\begin{bmatrix} 3/\sqrt{13} & -2/\sqrt{13} \\ 2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}$$

so the equations of the axes are $y = -3x/2$ and $y = 2x/3$, and the angle of rotation is $\cos^{-1}(3/\sqrt{13}) = 33.7°$. Following is the plot analogous to what we did in the example:



(3) The point of this problem is simply to warm up for quantum mechanics. We do

$$\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - (i)(-i) = \lambda^2 - 1 = 0$$

so the two eigenvalues are $\lambda = \pm 1$. For $\lambda = 1$ the eigenvector equation for the bottom component says $iv_x^{(1)} - v_y^{(1)} = 0$ so $v_y^{(1)} = iv_x^{(1)}$, and for $\lambda = -1$ the eigenvector equation for the bottom component says $iv_x^{(-1)} + v_y^{(-1)} = 0$ so $v_y^{(-1)} = -iv_x^{(-1)}$, giving the eigenvectors

$$\underline{v}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \underline{v}^{(-1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{with} \quad \tilde{\underline{v}}^{(1)}\,\underline{v}^{(-1)} = \frac{1}{2}\left[1(1) + (-i)(-i)\right] = 0$$

(4) Follow the standard procedure to get the characteristic equation and the eigenvalues.

$$\underline{\underline{A}} = \begin{vmatrix} 13-\lambda & 4 & -2 \\ 4 & 13-\lambda & -2 \\ -2 & -2 & 10-\lambda \end{vmatrix} = -\left[(\lambda-18)(\lambda-9)^2\right] = 0$$

where I used Simplify[Det[A -λ IdentityMatrix[3]]] to do the calculation. Therefore, the three eigenvalues are $\lambda^{(1)} = 18$ and $\lambda^{(2)} = 9 = \lambda^{(3)}$, that is, a repeating eigenvalue. For $\lambda = \lambda^{(1)}$,

$$\underline{\underline{A}} = \begin{bmatrix} -5 & 4 & -2 \\ 4 & -5 & -2 \\ -2 & -2 & -8 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thinking of this as three equations, it is easy to see the adding the first two and then multiplying by two gives the third equation. That is, the third equation is not independent of the other two, and we can write

$$\begin{aligned} -5v_1^{(1)} + 4v_2^{(1)} - 2v_3^{(1)} &= 0 \\ 4v_1^{(1)} - 5v_2^{(1)} - 2v_3^{(1)} &= 0 \end{aligned}$$

Subtract these two to get $-9v_1^{(1)} + 9v_2^{(1)} = 0$ or $v_2^{(1)} = v_1^{(1)}$, so $-v_1^{(1)} - 2v_3^{(1)} = 0$ and $v_3^{(1)} = -v_1^{(1)}/2$. The normalized eigenvector is therefore

$$\underline{v}^{(1)} = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

which is confirmed by MATHEMATICA. Now for $\lambda = \lambda^{(2)} = \lambda^{(3)} = 9$,

$$\underline{\underline{A}} = \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} v_1^{(\lambda)} \\ v_2^{(\lambda)} \\ v_3^{(\lambda)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all three equations are the same, namely $v_3^{(\lambda)} = 2v_1^{(\lambda)} + 2v_2^{(\lambda)}$. One approach is to set $v_2^{(\lambda)} = 0$ in which case $v_3^{(\lambda)} = 2v_1^{(\lambda)}$ and then $v_3^{(\lambda)} = 0$ in which case $v_2^{(\lambda)} = -v_1^{(\lambda)}$ and then normalize them. In fact this is what MATHEMATICA gives you, *but this does not result in orthogonal eigenvectors*. However, orthogonality can be added as another constraint, and this works because any vector $a\underline{v}^{(2)} + b\underline{v}^{(3)}$ would still have eigenvalue $\lambda = 9$. That is, orthogonality can get you to find a and b. This is enough for now, but it was worth the effort it took to get to this point. You'll see more of this when you study quantum mechanics.

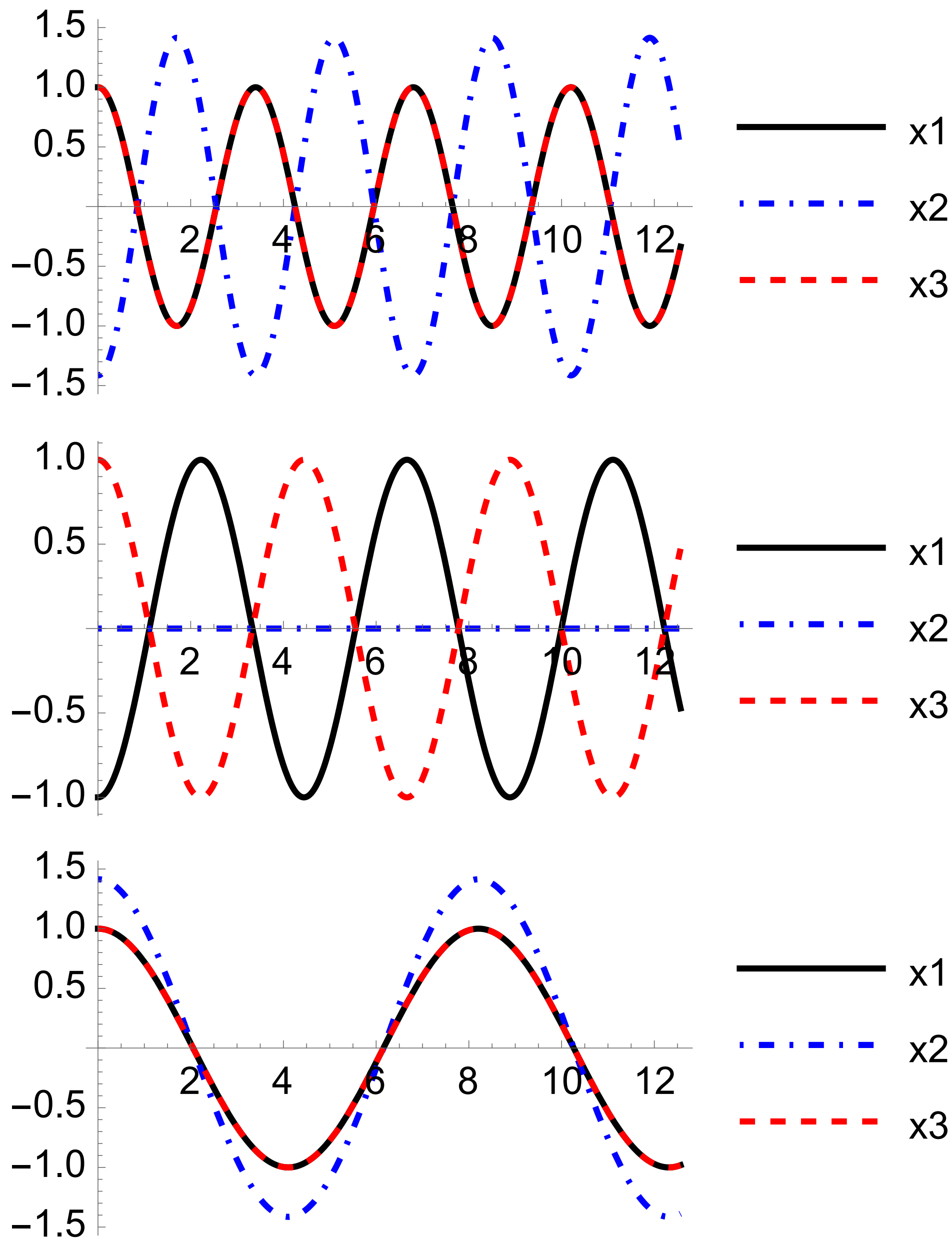
(5) It is straightforward to write down the three equations of motion, namely

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) = kx_1 - 2kx_2 + kx_3 \\ m\ddot{x}_3 &= -k(x_3 - x_2) - k - x_3 = kx_2 - 2kx_3 \end{aligned}$$

Defining $\omega_0^2 = k/m$ and setting $x_i = a_i e^{i\omega t}$ gives $\omega^2 \underline{a} = \omega_0^2 \underline{\underline{\Omega}}\, \underline{a}$ where

$$\underline{\underline{\Omega}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \text{and} \qquad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

which is an eigenvalue problem for the symmetric real matrix $\underline{\underline{\Omega}}$ where the eigenvalues are $\lambda = \omega^2/\omega_0^2$. For the rest, see the accompanying MATHEMATICA notebook. The eigenvalues are $\lambda^{(1)} = 2 + \sqrt{2}$, $\lambda^{(2)} = 2$, and $\lambda^{(3)} = 2 - \sqrt{2}$. Plots of the motions are below. In the high frequency mode (1), masses 1 and 3 are in phase with each other with the same amplitude, and the middle mass oscillates against them with a bit larger amplitude. In the middle frequency mode (2), mass 2 is stationary and masses 1 and 3 oscillation against each other, with the same amplitude. In low frequency mode (3), all three masses oscillate in phase, with the amplitude of mass 2 a bit larger than the amplitudes of masses 1 and 3.



PHYS2502 Mathematical Physics Homework #12 Due 12 Apr 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A particle of mass m moves in one dimension x over some time interval $t_1 \leq t \leq t_2$, under the influence of a force $F(x) = -dV/dx$, some function $V(x)$. Show that finding the function $x(t)$ which minimizes

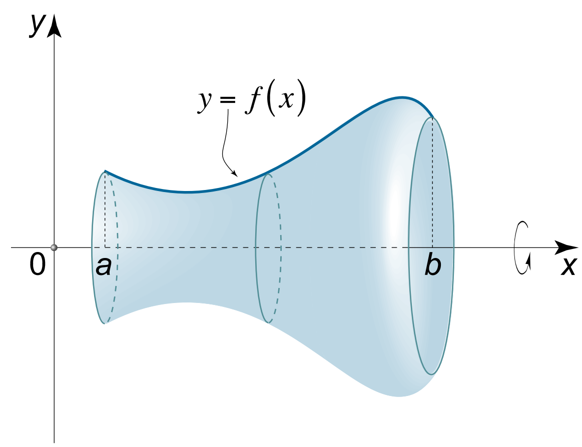
$$S = \int_{t_1}^{t_2} L(x, \dot{x})\, dt \qquad \text{where} \qquad L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x)$$

is the same as showing that $x(t)$ is determined by Newton's Second Law of Motion. Then show that since L does not explicitly depend on t, total mechanical energy is conserved.

(2) We discussed in class two special cases which allowed the Euler-Lagrange equations to be integrated once. In fact, when we first started talking about the Calculus of Variations, we found that a straight line gave the shortest distance between two points, using the first of these special cases. Show that the functional for the shortest path between two points is also an example of the second special case, and use that form to show that solution is a straight line. (As I write this, the second case is shown in Equation 7.7, Section 7.2.1 in the course notes.)

(3) Use MATHEMATICA to find and plot the brachistochrone solution for a bead starting at the origin and ending at $(a, b) = (1, 2)$.

(4) A "surface of revolution" is formed when a shape given by $y = f(x)$ for $a \leq x \leq b$ is rotated about the x-axis, as shown in the figure. Find the form of thefunction $f(x)$ which minimizes the surface area. You don't need to solve for the constants of integration in terms of the fixed points of $f(x)$ at $x = a$ and $x = b$.



(5) A chain of length $L > 2a$ hangs freely between two points $x = \pm a$ on the x-axis in the xy plane. Find the equation $f(x)$ that describes the resting shape of the chain, assuming that this is the shape that minimizes the center of gravity. Of course, the length L must remain fixed, and $f(\pm a) = 0$. Eliminate whichever constants of integration are easiest, but come up with a simple physical interpretation of the Lagrange multiplier used to set the length constraint.

(1) Just apply the Euler Lagrange equation where the independent variable is t and the dependent variable is $x(t)$. Then

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{dV}{dx} - \frac{d}{dt}m\dot{x} = F - m\ddot{x} = 0 \qquad \text{so} \qquad F = m\ddot{x} = ma$$

The "second special case" applies since $\partial L/\partial t = 0$, so

$$L - \dot{x}\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m\dot{x}^2 - V(x) - \dot{x}(m\dot{x}) = -\frac{1}{2}m\dot{x}^2 - V(x) = \text{constant} \equiv -E$$

(2) For the straight line, we apply the Euler Lagrange equation to

$$F[f'(x)] = \left[1 + (f'(x))^2\right]^{1/2}$$

which indeed does not depend explicitly on x. The special case therefore gives

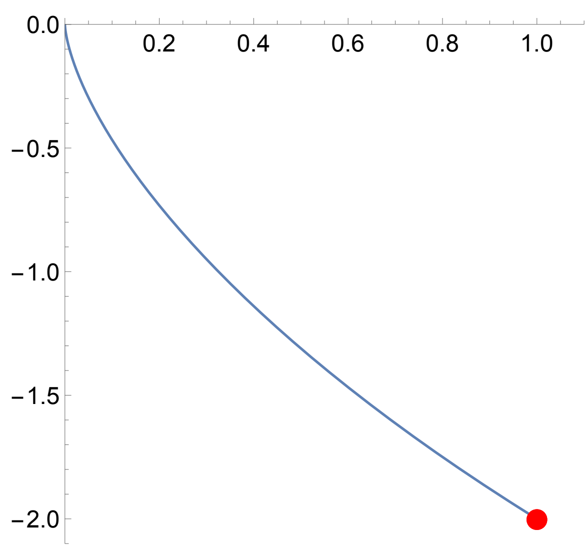
$$\begin{aligned} F - f'\frac{\partial F}{\partial f'} &= \left[1 + (f'(x))^2\right]^{1/2} - f'(x)\frac{f'(x)}{\left[1 + (f'(x))^2\right]^{1/2}} \\ &= \frac{1}{\left[1 + (f'(x))^2\right]^{1/2}}\left[1 + (f'(x))^2 - (f'(x))^2\right] = \frac{1}{\left[1 + (f'(x))^2\right]^{1/2}} = \text{constant} \end{aligned}$$

which implies that $f'(x)$ =constant, so, once again, the solution is a straight line.

(3) The equation of the cycloid is

$$\begin{aligned} x &= \frac{c}{2}(2\theta - \sin 2\theta) \\ y &= -c\sin^2\theta \end{aligned}$$

where it is easier to plot by letting y be negative going downward. Setting $\theta = 0$ gives the starting point at the origin. Setting $x = a$ and $y = (-)b$ gives two equations to solve for c and the ending value of $\theta = \theta_{\text{end}}$. Use $b = c\sin^2\theta$ to eliminate c in the first equation, and numerically find that value of θ_{end} that solves the second equation. See the accompanying MATHEMATICA notebook. For $(a, b) = (1, 2)$ I find $\theta_{\text{end}} = 0.70069$. Here is the plot:



(4) The radius of a thin "hoop" at horizontal position x is $y = f(x)$ and the width of the hoop is just ds, so adding up the hoop gives us the surface area

$$S = \int_a^b 2\pi y \, ds = 2\pi \int_a^b y \sqrt{1 + y'^2} \, dx$$

so we apply the Euler-Lagrange equation to $F(y, y') = y \sqrt{1 + y'^2}$. This is another example of the "second special case" since $\partial F / \partial x = 0$. Therefore

$$F - y' \frac{\partial F}{\partial y'} = y \sqrt{1 + y'^2} - yy' \frac{y'}{\sqrt{1 + y'^2}} = \frac{y}{\sqrt{1 + y'^2}} \left[1 + y'^2 - y'^2 \right] = \frac{y}{\sqrt{1 + y'^2}} = \text{constant}$$

Squaring both sides and rearranging a little gives

$$y^2 = c^2 (1 + y'^2) \qquad \text{or} \qquad \frac{dy}{dx} = \sqrt{\frac{y^2}{c^2} - 1} \qquad \text{or} \qquad \sqrt{\frac{c^2}{y^2 - c^2}} \, dy = dx$$

where c is some constant. The left side is integrated easily if $y = c \cosh t$. In this case we have $y^2 - c^2 = c^2 \sinh^2 t$ and $dy = c \sinh t \, dt$ so that

$$\sqrt{\frac{c^2}{c^2 \sinh^2 t}} \, c \sinh t \, dt = c \, dt = dx \qquad \text{so} \qquad x = ct + b$$

where b is the second constant of integration. This gives a neat form for $y = f(x)$, namely

$$y = f(x) = c \cosh \left(\frac{x - b}{c} \right)$$

(5) Let the chain have a linear mass density μ. If the chain hangs in the xy plane with $+y$ vertical, with shape $y = f(x)$, then the center of gravity is given by

$$S = \frac{1}{\mu L}\int_{-a}^{a} y\,\mu ds = \frac{1}{L}\int_{-a}^{a} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

In order to keep the length constant, we use a Lagrange multiplier λ with the integral of ds. We won't bother keeping the constant $1/L$ and write

$$\delta\int_{-a}^{a}\left\{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}+\lambda\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right\}dx = \delta\int_{-a}^{a}(y+\lambda)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 0$$

Note that, just as in the case of the maximum area under a flexible ' 'rope", the multiplier handles the difference in units of the two quantities. If we wanted to keep the L in the above expression, then the multiplier would be as a fraction of L.

This is, again, an example of special case #2, since there is no explicit x dependence, so

$$(y+\lambda)\sqrt{1+y'^2} - y'^2(y+\lambda)\frac{1}{\sqrt{1+y'^2}} = \frac{y+\lambda}{\sqrt{1+y'^2}} = \text{constant}$$

At this point, things look very similar to Problem (4). We write

$$(y+\lambda)^2 = c^2(1+y'^2) \qquad \text{so} \qquad \frac{dy}{dx} = \sqrt{\frac{(y+\lambda)^2}{c^2}-1}$$

so make the substitution $y+\lambda = c\cosh t$ and so $dy = c\sinh t\,dt$ and then

$$\sqrt{\frac{1}{\sinh^2 t}}\,c\sinh t\,dt = ct = dx \qquad \text{so} \qquad x = ct + b$$

and the equation describing the hanging chain is

$$y = c\cosh\left(\frac{x-b}{c}\right) - \lambda$$

Since $y = 0$ and $x = \pm a$ we have $c\cosh[(a \pm b)/c] = \lambda$ which can only be satisfied if $b = 0$. That is, the catenary is symmetric about the y-axis, as you would expect. This also tells us that $\lambda = c\cosh(a/c)$ so we could find c and λ after applying the length constraint and integrating the curve, but that's not important. What is clear, though, is that

$$y = c\cosh\left(\frac{x}{c}\right) - \lambda$$

so, physically, λ is the lowest hanging distance of the chain.

PHYS2502 Mathematical Physics Homework #13 Due 19 Apr 2022

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) This problem involves functions that are not analytic everywhere.

(a) A complex function $f(z) = 2y + ix$ where $z = x + iy$. Use the definition of the derivative directly to show that $f'(z)$ does not exist anywhere in the complex plane. Then show that this is consistent with the Cauchy-Riemann relations.This problem involves functions that are analytic.

(b) A complex function $f(z) = |x| - i|y|$ where $z = x + iy$. Where, if anywhere, in the complex plane is this function analytic?

(2) This problem involves functions that are analytic.

(a) Prove that $f(z) = e^z$ is analytic everywhere in the complex plane.

(b) Show that if $f(z)$ is an analytic function of z, then $g(z) = zf(z)$ is also an analytic function of z. Use this to explain why

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

is an analytic function of z.

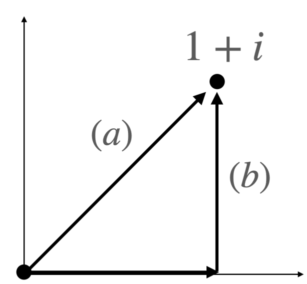
(3) Find an analytic function $f(z) = u(x, y) + iv(x, y)$ whose imaginary part is

$$v(x, y) = (y \cos y + x \sin y)e^x$$

(4) Referring to the diagram on the right, calculate the integral

$$\int_0^{1+i} (z^2 - z)\, dz$$

along the paths (a) and (b), where (b) is a horizontal step followed by a vertical step. Explain why the two results compare to each other the way that they do.



(5) Calculate the integral

$$\int_0^{\infty} \frac{1}{x^2 + 1} dx$$

in two different ways. First, use the substitution $x = \tan\theta$, and second as a contour integral in the complex plane. You can use MATHEMATICA to check your answer.

(1) (a) Write $\Delta z = \Delta x + i\Delta y$ with $\Delta y = a\Delta x$ for some real constant a, Then

$$f'(z) = \lim_{\Delta x \to 0} \frac{[2(y+\Delta y) + i(x+\Delta x)] - [2y + ix]}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{(2a+i)\Delta x}{(a+i)\Delta x} = \frac{2a+i}{a+i}$$

The result depends on the direction we approach z, that is the value of a, so the derivative doesn't exist at any z. With $u(x,y) = 2y$ and $v(x,y) = x$, the first Cauchy Riemann relation is satisfied as $0 = 0$, but the second is violated (for all z) wince $2 \neq -1$.

(b) The second Cauchy Riemann relation is satisfied for all z. However

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}|x| = +1 \quad x > 0 \quad \text{and} \quad = -1 \quad x < 0$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-|y|) = -1 \quad y > 0 \quad \text{and} \quad = +1 \quad y < 0$$

so the function is analytic in the two quadrants $(x > 0, y < 0)$ and $(x < 0, y > 0)$. Of course, these two regions correspond to $f(z) = z$ and $f(z) = -z$.

(2) (a) If we write $f(z) = e^x e^{iy}$ then $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$, so

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \text{and} \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

and the first Cauchy Riemann relation is satisfied. Also,

$$\frac{\partial v}{\partial x} = e^x \sin y \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

and the second is met as well. Therefore, the function is analytic for all z.

(b) Notationally, this is a little tricky. For $f = u + iv$, write $g = u_g + iv_g$ where $u_g = (xu - yv)$ and $v_g = (yu + xv)$. Our job is therefore to test the Cauchy Riemann relations on u_g and v_g when we know that they hold on u and v. We have

$$\begin{aligned}
\frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y} &= u + x\frac{\partial u}{\partial x} - y\frac{\partial v}{\partial x} - \left[u + y\frac{\partial u}{\partial y} + x\frac{\partial v}{\partial y}\right] \\
&= u + x\frac{\partial u}{\partial x} - y\frac{\partial v}{\partial x} - \left[u - y\frac{\partial v}{\partial x} + x\frac{\partial u}{\partial x}\right] = 0 \\
\frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y} &= y\frac{\partial u}{\partial x} + v + x\frac{\partial v}{\partial x} + \left[x\frac{\partial u}{\partial y} - v - y\frac{\partial v}{\partial y}\right] \\
&= y\frac{\partial u}{\partial x} + v + x\frac{\partial v}{\partial x} + \left[-x\frac{\partial v}{\partial x} - v - y\frac{\partial u}{\partial x}\right] = 0
\end{aligned}$$

so $g(z)$ is also analytic. Since $f(z) = 1$ is clearly analytic, just keep adding powers of z to show that $f(z) = z^n$ is analytic. Since the sum of two analytic functions is also analytic, just keep repeating the sum to show that the expansion in powers of z is analytic.

(3) We have to find a general solution to the partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &= (1 - y\sin y + x\cos y)e^x \\ \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &= -(y\cos y + x\sin y)e^x \end{aligned}$$

Integrate the first equation to get

$$u(x,y) = (1 - y\sin y + x\cos y)e^x + C$$

and it is clear that the second equation is satisfied for any value of C. So, we're done.

(4) The path (a) corresponds to $x(t) = t$ and $y(t) = t$ with $0 \le t \le 1$. Therefore

$$\begin{aligned} \int_0^{1+i} (z^2 - z)\,dz &= \int_0^1 [(x+iy)^2 - (x+iy)]\,(dx + i\,dy) \\ &= \int_0^1 [t^2(1+i)^2 - t(1+i)]\,(1+i)dt = (1+i)^2 \int_0^1 [(1+i)t^2 - t]\,dt \\ &= 2i\left[(1+i)\frac{1}{3} - \frac{1}{2}\right] = 2i\left[-\frac{1}{6} + i\frac{1}{3}\right] = -\frac{2}{3} - i\frac{1}{3} \end{aligned}$$

The path (b) is first along $y = 0$ for $0 \le x \le 1$, and then along $x = 1$ for $0 \le y \le 1$, so

$$\begin{aligned} \int_0^{1+i} (z^2 - z)\,dz &= \int_0^1 (x^2 - x)\,dx + \int_0^1 [(1+iy)^2 - (1+iy)]\,i\,dy \\ &= \left(\frac{1}{3} - \frac{1}{2}\right) + i\int_0^1 [1 + 2iy - y^2 - 1 - iy]\,dy \\ &= -\frac{1}{6} + i\int_0^1 [iy - y^2]\,dy = -\frac{1}{6} + i\left[i\frac{1}{2} - \frac{1}{3}\right] = -\frac{2}{3} - i\frac{1}{3} \end{aligned}$$

The two results are equal. But they had to be. The integrand is an analytic function, and the contour integral of an analytic function has to be independent of the path. This is a consequence of the Cauchy Goursat theorem, where the integral around a closed path for an analytic function is zero. Just reverse, for example, path (a), and add it to (b), and you have a closed path. Reversing (a) reverses the sign of the integral, so this implies that the two integrals above have to be the same.

(5) Mathematica tells us that the integral equals $\pi/2$. Putting $x = \tan\theta = \sin\theta/\cos\theta$ gives $dx = [\cos\theta/\cos\theta - \sin\theta/(\cos^2\theta)(-\sin\theta)]d\theta = 1 + \sin^2\theta/\cos^2\theta = d\theta/\cos^2\theta$, and so

$$\int_0^\infty \frac{1}{x^2+1}dx = \int_0^{\pi/2} \frac{1}{\tan^2\theta + 1}\frac{d\theta}{\cos^2\theta} = \int_0^{\pi/2} \cos^2\theta \frac{d\theta}{\cos^2\theta} = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

On the other hand, we can write this as a closed contour integral, namely

$$\int_0^\infty \frac{1}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{x^2+1}dx = \frac{1}{2}\oint_C \frac{1}{z^2+1}dz = \frac{1}{2}\oint_C \frac{1}{(z+i)(z-i)}dz$$

where the contour C runs along the x-axis and closes either counter-clockwise in the upper half of the complex plane, or clockwise in the lower half. In the former case, we pick up the pole at $z = +i$ and a factor of $2i\pi$, whereas in the latter case, we pick up the pole at $z = -i$ and a factor of $-2i\pi$. Therefore we find

$$\int_0^\infty \frac{1}{x^2+1}dx = \frac{1}{2}(\pm 2i\pi)\frac{1}{\pm i \pm i} = \frac{\pi}{2}$$