PHYS2502 Mathematical Physics
Spring 2022

Final Exam
Thursday 28 Apr 2022

There are **five questions** and you are to work all of them. You are welcome to use the posted class notes, your own notes, or any other resources, but you may not communicate with another human. Of course, if you have questions, you are encouraged to ask the person proctoring the exam.

The five problems will be equally weighted. If you are stuck on one, move on to another and come back if you have time.

Please start each problem on a new page in your exam booklet.

Good luck!

(1) An object of mass m moves in one dimension x and is subjected to a force $F(t) = F_0 e^{-\alpha t}$ where F_0 and α are positive constants. Write the second order differential equation that governs the motion, and solve it to find the position $x(t)$ as a function of time if the particle starts from rest at $x = 0$.

(2) Prove the following true statement:

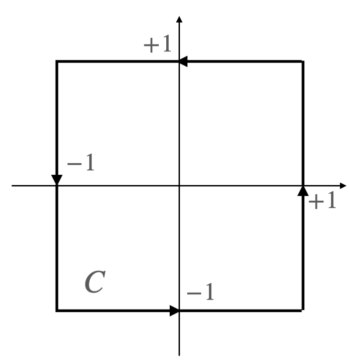
$$\frac{\pi^2}{3!} - \frac{\pi^4}{5!} + \frac{\pi^6}{7!} - \frac{\pi^8}{9!} + \cdots = 1$$

(3) A set of points (x, y) in a plane is given by $5x^2 - 4xy + 2y^2 = 30$ where x and y are orthogonal coordinates. Write this equation as $ax'^2 + by'^2 = 30$, with numerical values for a and b, where the x'- and y'-axes are rotated with respect to the x- and y-axes. Find the angle of rotation.

(4) A force field $\vec{F}(\vec{r})$ is said to be "conservative" if the work $W = \int_1^2 \vec{F}(\vec{r}) \cdot d\vec{r}$ done between any two points in space 1 and 2 is independent of the path taken between these two points. Show that any force field of the form $\vec{F}(\vec{r}) = f(r)\hat{r}$, where r is the distance from the origin and $\hat{r}$ is the radial unit vector, is conservative.

(5) For the closed contour C shown on the right in the complex plane, a square of side length 2 centered on the origin, calculate the integral

$$\oint_C \frac{z^2}{2z - 1}\, dz$$



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Solutions

(1) The differential equation is $mx''(t) = F(t) = F_0 e^{-\alpha t}$, which does not depend on $x(t)$ or $x'(t)$, and the initial conditions are $x(0) = 0$ and $x'(0) = 0$. Solving first for $v(t) = x'(t)$,

$$v'(t) = \frac{F_0}{m} e^{-\alpha t} \qquad \text{so} \qquad v(t) = x'(t) = c - \frac{F_0}{m\alpha} e^{-\alpha t} \qquad \text{with} \qquad v(0) = c - \frac{F_0}{m\alpha} = 0$$

so $c = F_0/m\alpha$. Now we can solve for $x(t)$ from

$$x'(t) = \frac{F_0}{m\alpha} - \frac{F_0}{m\alpha} e^{-\alpha t} \qquad \text{so} \qquad x(t) = \frac{F_0}{m\alpha} t + \frac{F_0}{m\alpha^2} e^{-\alpha t} + c \qquad \text{with} \qquad x(0) = \frac{F_0}{m\alpha^2} + c = 0$$

so $c = -F_0/m\alpha^2$ and the position as a function of time is

$$x(t) = \frac{F_0}{m\alpha^2}(\alpha t + e^{-\alpha t} - 1)$$

(2) This just follows from the series expansion for $\sin \pi = 0$ and dividing through by π, i.e.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots \\ \text{so} \qquad \sin \pi &= \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} + \cdots = 0 \\ \text{and then} \qquad 1 &= \frac{\pi^2}{3!} - \frac{\pi^4}{5!} + \frac{\pi^6}{7!} - \frac{\pi^8}{9!} + \cdots \end{aligned}$$

(3) The values of a and b are just the eigenvalues of the symmetric matrix that makes up the left hand side of the equation. Therefore we need to write

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = 10 - 7\lambda + \lambda^2 - 4 = \lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1) = 0$$

so the eigenvalues are $\lambda = 6$ and $\lambda = 1$. If we take the "first" eigenvalue to be $\lambda = 6$, and the "second" to be $\lambda = 1$, then the rotated equation will be

$$6x'^2 + y'^2 = 30$$

To get the rotation angle, we have to consider the rotation matrix, which is made up column-by-column from the eigenvectors. It should be clear that

$$-u_1^{(6)} - 2u_2^{(6)} = 0 \qquad \text{and} \qquad -2u_1^{(1)} + u_2^{(1)} = 0$$

so that the normalized eigenvectors are

$$\underline{u}^{(6)} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \qquad \text{and} \qquad \underline{u}^{(1)} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

so the rotation angle is

$$\theta = \cos^{-1} \frac{2}{\sqrt{5}} = 0.464 = 26.6^\circ$$

We can check that we did this all correctly by the calculation

$$
\begin{aligned}
& \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \\
= & \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 12/\sqrt{5} & 1/\sqrt{5} \\ -6/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} (24+6)/5 & 0 \\ 0 & (1+4)/5 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

(4) If the work done between the two paths is the same, then I can reverse the direction of one of the paths and have an integral around a closed loop that is zero. Therefore, by Stokes' theorem, the force will be conservative if its curl is zero. In cylindrical coordinates, the curl is

$$\vec{\nabla} \times \vec{V} = \hat{r} \left(\frac{1}{r} \frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z} \right) + \hat{\phi} \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \right) + \hat{k} \frac{1}{r} \left(\frac{\partial}{\partial r} r V_\phi - \frac{\partial V_r}{\partial \phi} \right)$$

In our case, the vector field $\vec{F}$ only has a radial component, and this radial component depends only on r and not on ϕ or z, so it is clear that the curl is zero.

(5) This is a simple application of the Cauchy integral theorem.

$$\oint_C \frac{z^2}{2z-1}\, dz = \oint_C \frac{z^2/2}{z-1/2}\, dz = 2\pi i \frac{1}{2} \left(\frac{1}{2} \right)^2 = \frac{\pi i}{4}$$