Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #1 19 Jan 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Given the two complex numbers

$$z_1 = 1 + i \qquad \text{and} \qquad z_2 = 3 - 4i$$

find the following:

(a) z_1^2
(b) z_1^*
(c) $z_1 z_2^*$
(d) $|z_1|$ and $|z_2|$
(e) $|z_1 z_2|$

PHY2502 Mathematical Physics Quiz #1 Solution

Given the two complex numbers

$$z_1 = 1 + i \qquad \text{and} \qquad z_2 = 3 - 4i$$

find the following:

(a) z_1^2
(b) z_1^*
(c) $z_1 z_2^*$
(d) $|z_1|$ and $|z_2|$
(e) $|z_1 z_2|$

$$z_1^2 = (1+i)^2 = 1 + 2i - 1 = 2i$$
$$z_1^* = 1 - i$$
$$z_1 z_2^* = (1+i)(3+4i) = 3 + 3i + 4i - 4 = -1 + 7i$$
$$|z_1| = \sqrt{1+1} = \sqrt{2} \qquad \text{and} \qquad |z_2| = \sqrt{9+16} = 5$$

$$\begin{aligned} |z_1 z_2| &= |(1+i)(3-4i)| = |3 + 3i - 4i + 4| = |7 - i| = \sqrt{49+1} = 5\sqrt{2} \\ &= |z_1||z_2| \end{aligned}$$

Name: ____________________

PHYS2502 Mathematical Physics S23 Quiz #2 26 Jan 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

A pendulum is made from a mass m hanging from a (massless) string of length ℓ. The motion of the string is dictated by the acceleration g due to gravity. Use dimensional analysis to find the relevant time scale τ for the pendulum motion. Compare this to what you know to be the period of a pendulum undergoing small amplitude oscillations.

PHY2502 Mathematical Physics Quiz #2 Solution

A pendulum is made from a mass m hanging from a (massless) string of length ℓ. The motion of the string is dictated by the acceleration g due to gravity. Use dimensional analysis to find the relevant time scale τ for the pendulum motion. Compare this to what you know to be the period of a pendulum undergoing small amplitude oscillations.

$$\tau = m^x \ell^y g^z$$
$$[\tau] = [m]^x [\ell]^y [g]^z$$
$$T = M^x L^y L^z T^{-2z} = M^x L^{y+z} T^{-2z}$$

Therefore $x = 0$, $y + z = 0$, $z = -1/2$, and so $y = -z = +1/2$. In other words

$$\tau = \ell^{1/2} g^{-1/2} = \sqrt{\frac{\ell}{g}}$$

This agrees with what you know from your introductory physics course, namely the period of a pendulum $T = 2\pi\sqrt{\ell/g}$. Dimensional analysis can't, of course, predict the factor of 2π.

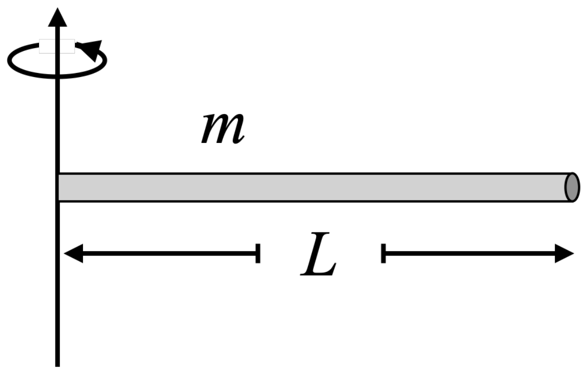
Name: ____________________

PHYS2502 Mathematical Physics S23 Quiz #3 2 Feb 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

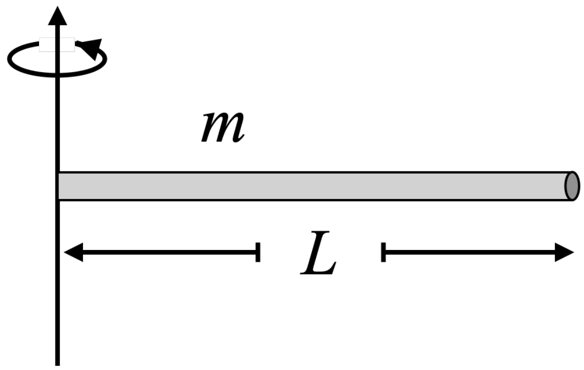
Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

You know that the moment inertia of an object with mass m rotating about a specific axis is $I = \int dm\, r^2$ where r is the distance from the axis of the infinitesimal mass element dm, and the integral is over the entire object. Use this to determine the moment of inertia of a long, thin rod of mass m, length L, and uniform linear mass density λ, rotating about an axis at one end of and perpendicular to the rod, as shown here:



PHY2502 Mathematical Physics Quiz #3 Solution

You know that the moment inertia of an object with mass m rotating about a specific axis is $I = \int dm\, r^2$ where r is the distance from the axis of the infinitesimal mass element dm, and the integral is over the entire object. Use this to determine the moment of inertia of a long, thin rod of mass m, length L, and uniform linear mass density λ, rotating about an axis at one end of and perpendicular to the rod, as shown here:



$$I = \int_0^L dm\, r^2 = \int_0^L \lambda\, dr\, r^2 = \int_0^L \frac{m}{L} r^2\, dr = \frac{m}{L}\left.\frac{r^3}{3}\right|_0^L = \frac{m}{L}\frac{L^3}{3} = \frac{1}{3} mL^2$$

Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #4 9 Feb 2023

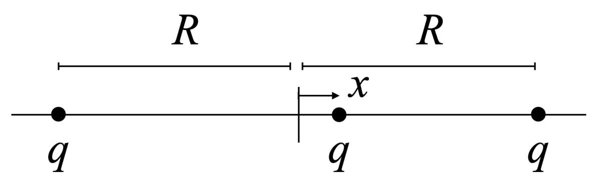
You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

You recall from some physics course that the force between two electric charges q_1 and q_2 is

$$F = k\frac{q_1 q_2}{d^2}$$

where k is a constant and d is the distance between the two charges. The force is repulsive if the signs of q_1 and q_2 are the same. Now consider the case of three identical charges q constrained to be on a line as shown here:



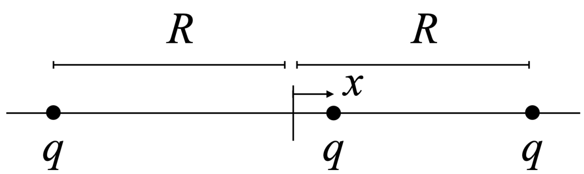
The outer two charges are fixed to the line, and are separated by a distance $2R$. The third charge can move along the line, where x measures its distance from the midpoint. Find the force on the middle charge in the limit $x \ll R$ and use this to find the (angular) frequency "ω" for small oscillations about the midpoint if the middle charge has mass m.

PHY2502 Mathematical Physics Quiz #4 Solution

You recall from some physics course that the force between two electric charges q_1 and q_2 is

$$F = k\frac{q_1 q_2}{d^2}$$

where k is a constant and d is the distance between the two charges. The force is repulsive if the signs of q_1 and q_2 are the same. Now consider the case of three identical charges q constrained to be on a line as shown here:



The outer two charges are fixed to the line, and are separated by a distance $2R$. The third charge can move along the line, where x measures its distance from the midpoint. Find the force on the middle charge in the limit $x \ll R$ and use this to find the (angular) frequency "ω" for small oscillations about the midpoint if the middle charge has mass m.

The force on the middle charge is positive from the charge on the left, and negative from the charge on the right, so

$$\begin{aligned} F = k\frac{q^2}{(R+x)^2} - k\frac{q^2}{(R-x)^2} &= k\frac{q^2}{R^2}\left[\frac{1}{(1+x/R)^2} - \frac{1}{(1-x/R)^2}\right] \\ &\approx k\frac{q^2}{R^2}\left[1 - 2\frac{x}{R} - \left(1 + 2\frac{x}{R}\right)\right] = -4k\frac{q^2}{R^3}x \end{aligned}$$

This has the form $F = -kx$ where $k \to 4kq^2/R^3$ is the stiffness of the spring, so

$$\omega^2 = \frac{“k”}{m} = k\frac{q^2}{mR^3} \qquad \text{and} \qquad \omega = \sqrt{k\frac{q^2}{mR^3}}$$

We can easily check the dimensionality of this result. The factor kq^2/R^2 has dimensions of force, namely MLT^{-2}. Dividing by mR leaves us with T^{-2}, and taking the square root indeed gives T^{-1} which is frequency.

Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #5 16 Feb 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Solve the following differential equation and boundary condition for the function $y(x)$:

$$(2x + 3)\, dx - 2y\, dy = 0 \qquad \text{and} \qquad y(0) = 1$$

PHY2502 Mathematical Physics Quiz #5 Solution

Solve the following differential equation and boundary condition for the function $y(x)$:

$$(2x+3)\,dx - 2y\,dy = 0 \qquad \text{and} \qquad y(0) = 1$$

This equation is both exact and separable, but the simple way to solve it is just to integrate through. Just doing it with indefinite integrals, we get

$$\int (2x+3)\,dx - \int 2y\,dy = x^2 + 3x - y^2 = C$$

Setting $x = 0$ and $y = 1$ gives $C = -1$ so

$$y^2 = x^2 + 3x + 1 \qquad \text{and} \qquad y(x) = \sqrt{x^2 + 3x + 1}$$

where we take the positive sign of the square root in order to satisfy $y(0) = +1$.

Name: ____________________

PHYS2502 Mathematical Physics S23 Quiz #6 23 Feb 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Using an expansion of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, find a recursion relation for the a_n so that this series is a solution to the differential equation

$$y''(x) - xy'(x) - y(x) = 0$$

Show that you get two different solutions which you can argue are independent.

For double credit, show that one of these solutions can be written in a simple, closed analytic form. You can check your answer with direct substitution, if you like.

PHY2502 Mathematical Physics Quiz #6 Solution

Using an expansion of the form $y(x)=\sum_{n=0}^{\infty}a_nx^n$, find a recursion relation for the a_n so that this series is a solution to the differential equation

$$y''(x)-xy'(x)-y(x)=0$$

Show that you get two different solutions which you can argue are independent.

For double credit, show that one of these solutions can be written in a simple, closed analytic form. You can check your answer with direct substitution, if you like.

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_nx^n \\
xy'(x) &= x\sum_{n=0}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} na_nx^n \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \\
0 &= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-na_n-a_n\right]x^n = \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n+1)a_n\right]x^n \\
a_{n+2} &= \frac{n+1}{(n+2)(n+1)}a_n = \frac{1}{n+2}a_n
\end{aligned}$$

One solution has $a_0=1$ and $a_1=0$, so just even powers of x, and the other has $a_0=0$ and $a_1=1$, so just odd powers of x. Just as we argued for the Hermite polynomials, these two power series would be independent of each other.

Now if we consider the even powers and write $n=2k$ where now $k=0,1,2,\ldots$, then $a_{k+1}=1/(2k+2)a_k=1/(k+1)(1/2)a_k$. Setting $a_0=1$, we get

$$a_1=\frac{1}{1}\frac{1}{2}, \qquad a_2=\frac{1}{2}\frac{1}{2}a_1=\frac{1}{2!}\frac{1}{2^2}, \qquad \text{so} \qquad a_k=\frac{1}{k!}\frac{1}{2^k}$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n=\sum_{k=0}^{\infty}a_kx^{2k}=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{2^k}x^{2k}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x^2}{2}\right)^k=e^{x^2/2}$$

It is worth checking to make sure $y(x)=e^{x^2/2}$ solves the differential equation.

$$\begin{aligned}
y''(x)-xy'(x)-y(x) &= \frac{d}{dx}(xe^{x^2/2})-x^2e^{x^2/2}-e^{x^2/2} \\
&= e^{x^2/2}+x^2e^{x^2/2}-x^2e^{x^2/2}-e^{x^2/2}=0
\end{aligned}$$

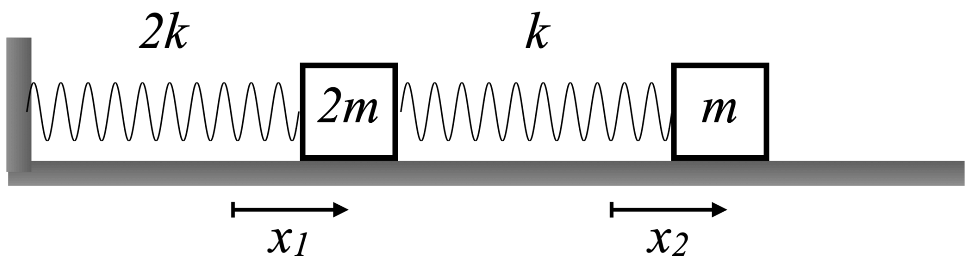
Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #7 2 Mar 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

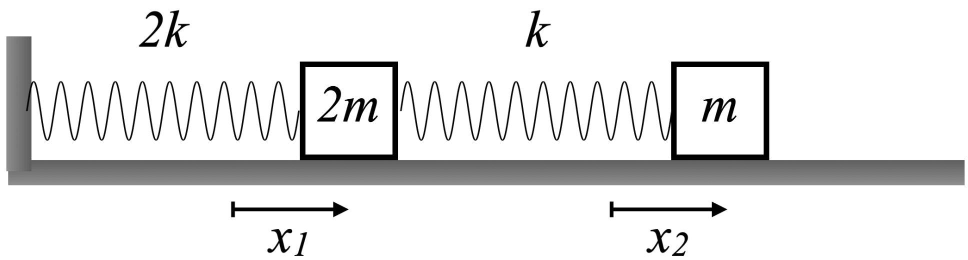
Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the two normal mode frequencies, that is, the two eigenvalues ω^2, for the coupled system of two masses and two springs shown below:



PHY2502 Mathematical Physics Quiz #7 Solution

Find the two normal mode frequencies, that is, the two eigenvalues ω^2, for the coupled system of two masses and two springs shown below:



Newton's Second Law for each of the two masses gives

$$
\begin{aligned}
2m\ddot{x}_1 &= -2kx_1 + k(x_2 - x_1) = -3kx_1 + kx_2 \\
m\ddot{x}_2 &= -k(x_2 - x_1) = kx_1 - kx_2
\end{aligned}
$$

Divide through by m, define $\omega_0^2 \equiv k/m$, and insert $x_{1,2} = a_{1,2}e^{i\omega t}$ to get

$$
\begin{aligned}
-2\omega^2 a_1 &= -3\omega_0^2 a_1 + \omega_0^2 a_2 \\
-\omega^2 a_2 &= \omega_0^2 a_1 - \omega_0^2 a_2
\end{aligned}
$$

Rearrange so that it looks like two equations for a_1 and a_2, that is

$$
\begin{aligned}
(3\omega_0^2 - 2\omega^2)a_1 - \omega_0^2 a_2 &= 0 \\
-\omega_0^2 a_1 + (\omega_0^2 - \omega^2)a_2 &= 0
\end{aligned}
$$

Set the determinant equal to zero and come up with a quadratic equation for ω^2.

$$(3\omega_0^2 - 2\omega^2)(\omega_0^2 - \omega^2) - \omega_0^4 = 2\omega^4 - 5\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

Finally, solve this equation for the two values of ω^2.

$$\omega^2 = \frac{5\omega_0^2 \pm \sqrt{25\omega_0^4 - 16\omega_0^4}}{4} = \frac{5 \pm 3}{4}\omega_0^2 \qquad = \qquad 2\omega_0^2 \qquad \text{and} \qquad \frac{1}{2}\omega_0^2$$

Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #8 16 Mar 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Two different straight lines containing points $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ are given by the equations

$$\vec{r} = \vec{m}_1 t + \vec{r}_0 \qquad \text{and} \qquad \vec{r} = \vec{m}_2 t + \vec{r}_0$$

where $\vec{m}_1 = -2\hat{i} + 3\hat{j} - 2\hat{k}$, $\vec{m}_2 = -2\hat{i} - 3\hat{j} + 4\hat{k}$, and $\vec{r}_0 = \hat{i} + \hat{j} + \hat{k}$. Find the equation of the plane that contains both of these lines, in the form $Ax + By + Cz = D$.

PHY2502 Mathematical Physics Quiz #8 Solution

Two different straight lines containing points $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ are given by the equations

$$\vec{r} = \vec{m}_1 t + \vec{r}_0 \qquad \text{and} \qquad \vec{r} = \vec{m}_2 t + \vec{r}_0$$

where $\vec{m}_1 = -2\hat{i} + 3\hat{j} - 2\hat{k}$, $\vec{m}_2 = -2\hat{i} - 3\hat{j} + 4\hat{k}$, and $\vec{r}_0 = \hat{i} + \hat{j} + \hat{k}$. Find the equation of the plane that contains both of these lines, in the form $Ax + By + Cz = D$.

The equation for a plane is $(\vec{r} - \vec{r}_0) \cdot \hat{n} = 0$ where $\vec{r}_0$ is a point in the plane and $\hat{n}$ is a normal (unit) vector. Both lines pass through $\vec{r}_0 = \hat{i} + \hat{j} + \hat{k}$, so we can use that as the point in the plane. The normal vector $\vec{n}$ has to be normal to both the direction of line #1 and line #2, so we can get it from the cross product of $\vec{m}_1$ and $\vec{m}_2$, namely

$$\begin{aligned} \vec{n} &= \vec{m}_1 \times \vec{m}_2 = (m_{1_y} m_{2_z} - m_{1_z} m_{2_y})\hat{i} + (m_{1_z} m_{2_x} - m_{1_x} m_{2_z})\hat{j} + (m_{1_x} m_{2_y} - m_{1_y} m_{2_x})\hat{k} \\ &= (12 - 6)\hat{i} + (4 + 8)\hat{j} + (6 + 6)\hat{k} = 6(\hat{i} + 2\hat{j} + 2\hat{k}) \end{aligned}$$

It's not necessary to normalize the normal vector, since you can just multiply through in the equation of plane by some constant, but just to be complete

$$|\vec{n}| = 6\sqrt{1 + 4 + 4} = 18 \qquad \text{so} \qquad \hat{n} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

To get to the standard form of a plane, we write $\vec{r} \cdot \hat{n} = \vec{r}_0 \cdot \hat{n}$, so

$$\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z = \frac{1}{3} + \frac{2}{3} + \frac{2}{3} = \frac{5}{3}$$

It's a little neater to multiply through by 3, that is

$$x + 2y + 2z = 5$$

Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #9 23 Mar 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

This quiz has **two parts**. Please answer both of them.

(a) An electric field in the x-direction permeates all space, and is of the form

$$\vec{E}(x,y,z) = E_0 \left(\frac{x}{L}\right)^2 \hat{i}$$

Find the amount of electric charge contained in a cube of side length L in the first octant, that is $x > 0$, $y > 0$, and $z > 0$, with a corner at the origin.

(b) Repeat this calculation, but now the electric field is

$$\vec{E}(x,y,z) = E_0 \left(\frac{x}{L}\right)^2 \hat{j}$$

PHY2502 Mathematical Physics Quiz #9 Solution

This quiz has **two parts**. Please answer both of them.

(a) An electric field in the x-direction permeates all space, and is of the form

$$\vec{E}(x,y,z) = E_0 \left(\frac{x}{L}\right)^2 \hat{\imath}$$

Find the amount of electric charge contained in a cube of side length L in the first octant, that is $x > 0$, $y > 0$, and $z > 0$, with a corner at the origin.

(b) Repeat this calculation, but now the electric field is

$$\vec{E}(x,y,z) = E_0 \left(\frac{x}{L}\right)^2 \hat{\jmath}$$

(a) Gauss' Law, with Gauss' Theorem, says that

$$Q_{\text{enc}} = k \oint_S \vec{E} \cdot d\vec{S} = k \int_V \vec{\nabla} \cdot \vec{E}\, dV$$

where $k = 1/4\pi$ in CGS units or $k = \epsilon_0$ in SI units. For the surface integral, realize that $\vec{E}$ only has a nonzero dot product along the faces parallel to the yz plane, and $x = 0$ on one of them, and $x = L$ on the other, so only one face contributes. The result is

$$\oint_S \vec{E} \cdot d\vec{S} = E_0 \left(\frac{L}{L}\right)^2 L^2 = E_0 L^2$$

If instead you want to use Gauss' theorem, we have

$$\int_V \vec{\nabla} \cdot \vec{E}\, dV = \int_0^L dx \int_0^L dy \int_0^L dz\, \frac{E_0}{L^2} 2x =, \frac{E_0}{L^2} \int_0^L 2x\, dx\, L\, L = E_0 L^2$$

which is, of course, the same as using the surface integral. Thereforre

$$Q_{\text{enc}} = \frac{1}{4\pi} E_0 L^2 \text{ (CGS)} \qquad \text{or} \qquad Q_{\text{enc}} = \epsilon_0 E_0 L^2 \text{ (SI)}$$

(b) The integrals are zero, so the enclosed charge is zero. For the surface integral, you integrate over the two faces parallel to the xz plane, but you get the same answer with opposite signs on each. For the volume integral, it is simple to see that $\vec{\nabla} \cdot \vec{E} = 0$.

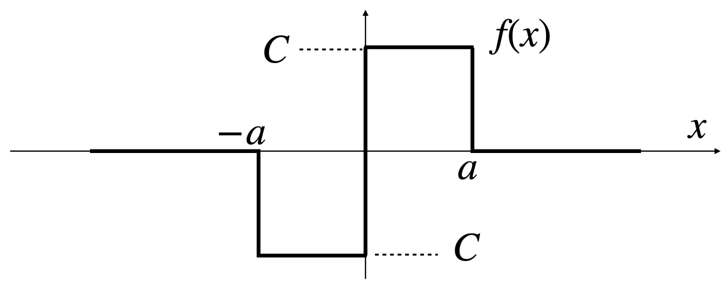
Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #10 30 Mar 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

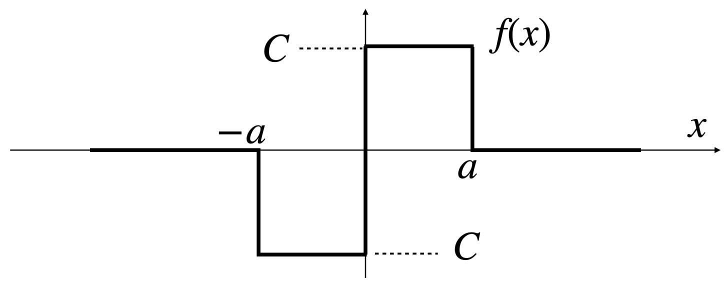
Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the Fourier Transform $A(k)$ for the pulse shape below, where C and a are constants:



PHY2502 Mathematical Physics Quiz #10 Solution

Find the Fourier Transform $A(k)$ for the pulse shape below, where C and a are constants:



$$
\begin{aligned}
A(k) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = \int_{-\infty}^{\infty} [\cos(kx) - i\sin(kx)] f(x)\, dx \\
&= -i \int_{-a}^{a} \sin(kx) f(x)\, dx \\
&= -2i \int_{0}^{a} C \sin(kx)\, dx \\
&= 2iC\, \frac{1}{k} \cos(kx)\Big|_0^a \\
&= 2iC \frac{1}{k} [\cos(ka) - 1] \\
&= -4iC \frac{1}{k} \sin^2\left(\frac{ka}{2}\right)
\end{aligned}
$$

Alternatively, we can just integrate the exponential:

$$
\begin{aligned}
A(k) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = \int_{-a}^{0} (-C) e^{-ikx}\, dx + \int_{0}^{a} (C) e^{-ikx}\, dx \\
&= -C\, \frac{1}{-ik} e^{-ikx}\Big|_{-a}^{0} + C\, \frac{1}{-ik} e^{-ikx}\Big|_{0}^{a} \\
&= \frac{C}{ik}\left(1 - e^{ika}\right) - \frac{C}{ik}\left(e^{-ika} - 1\right) \\
&= \frac{C}{ik}\left(2 - e^{ika} - e^{-ika}\right) \\
&= 2iC \frac{1}{k} [\cos(ka) - 1] \\
&= -4iC \frac{1}{k} \sin^2\left(\frac{ka}{2}\right)
\end{aligned}
$$

Name: ____________________

PHYS2502 Mathematical Physics S23 Quiz #11 6 Apr 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

For the two matrices

$$\underline{\underline{L}}_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \underline{\underline{L}}_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

(a) Determine if either, neither, or both of these matrices are Hermitian.

(b) Find the matrix given by $\underline{\underline{L}}_x \underline{\underline{L}}_y - \underline{\underline{L}}_y \underline{\underline{L}}_x$.

PHY2502 Mathematical Physics Quiz #11 Solution

For the two matrices

$$\underline{\underline{L}}_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \underline{\underline{L}}_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

(a) Determine if either, neither, or both of these matrices are Hermitian.

(b) Find the matrix given by $\underline{\underline{L}}_x\underline{\underline{L}}_y - \underline{\underline{L}}_y\underline{\underline{L}}_x$.

Taking the transpose and complex conjugate you get the same matrix back for both $\underline{\underline{L}}_x$ and $\underline{\underline{L}}_y$, so both of these matrices are Hermitian.

$$\begin{aligned}
\underline{\underline{L}}_x\underline{\underline{L}}_y &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix} \\
\underline{\underline{L}}_y\underline{\underline{L}}_x &= \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{bmatrix} \\
\text{so} \qquad \underline{\underline{L}}_x\underline{\underline{L}}_y - \underline{\underline{L}}_y\underline{\underline{L}}_x &= \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}
\end{aligned}$$

Name: ________________________________

PHYS2502 Mathematical Physics S23 Quiz #12 13 Apr 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Find the eigenvalues of the matrix

$$\underline{\underline{L}}_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

Extra Credit: Find the normalized eigenvectors corresponding to each of the three eigenvalues. Each correct answer is worth three points, but you get ten points for all three correct.

PHY2502 Mathematical Physics Quiz #12 Solution

Find the eigenvalues of the matrix

$$\underline{\underline{L}}_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

Extra Credit: Find the normalized eigenvectors corresponding to each of the three eigenvalues. Each correct answer is worth three points, but you get ten points for all three correct.

$$\begin{vmatrix} -\lambda & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & -\lambda & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & -\lambda \end{vmatrix} = -\lambda^3 - 2(-\lambda)\left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right) = -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = 0$$

so the three eigenvalues are $\lambda = 0$, $\lambda = +1$, and $\lambda = -1$.

For $\lambda = 0$, the first and third equations say that the middle component of the eigenvector must be zero. The second equation says that the first and third components are equal, so

$$v^{(\lambda=0)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = +1$, the second equation says the middle component os $i/\sqrt{2}$ times the first minus the third. Adding the first and third equations says that the first and third components are negatives of each other. Therefore

$$v^{(\lambda=+1)} = \frac{1}{2} \begin{bmatrix} 1 \\ 2i/\sqrt{2} \\ -1 \end{bmatrix}$$

For $\lambda = -1$, the second equation says the middle component os $i/\sqrt{2}$ times the third minus the first. Adding the first and third equations says that the first and third components are negatives of each other. Therefore

$$v^{(\lambda=-1)} = \frac{1}{2} \begin{bmatrix} 1 \\ -2i/\sqrt{2} \\ -1 \end{bmatrix}$$

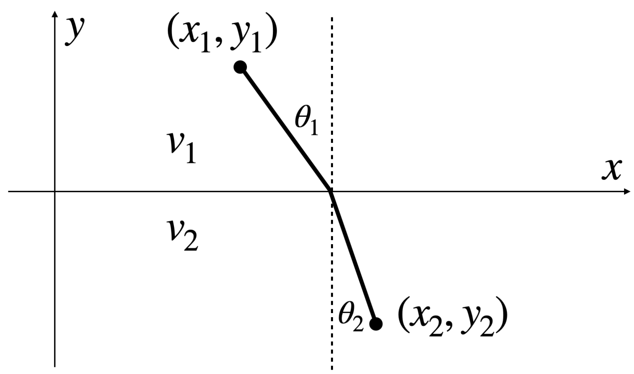
Name: ______________________________

PHYS2502 Mathematical Physics S23 Quiz #13 20 Apr 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

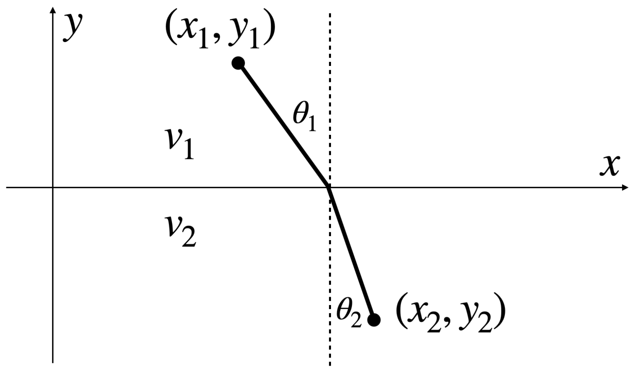
A lifeguard stands on a beach at point (x_1, y_1) and spots a swimmer in trouble in the water at point (x_2, y_2). She runs on the sand with speed v_1, and swims with a speed v_2. To save the swimmer, she wants to minimize the time it takes to get to him.



If the coastline coincides with the x-axis above, find the path of least time. Express your answer in terms of the angles θ_1 and θ_2 and the speeds v_1 and v_2. *Hints: The result should look familiar to you.* *Do not try to solve this problem using the Euler-Lagrange equation.*

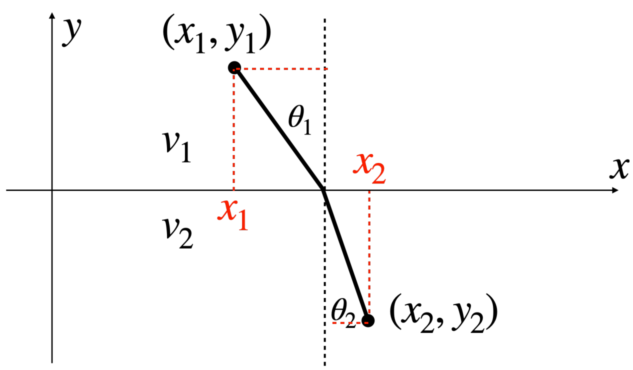
PHY2502 Mathematical Physics Quiz #13 <u>Solution</u>

A lifeguard stands on a beach at point (x_1, y_1) and spots a swimmer in trouble in the water at point (x_2, y_2). She runs on the sand with speed v_1, and swims with a speed v_2. To save the swimmer, she wants to minimize the time it takes to get to him.



If the coastline coincides with the x-axis above, find the path of least time. Express your answer in terms of the angles θ_1 and θ_2 and the speeds v_1 and v_2. *Hints: The result should look familiar to you. <u>Do not try to solve this problem using the Euler-Lagrange equation.</u>*

Refer to this annotated version of the figure:



The time it takes to get from point 1 to point 2 is

$$T(x) = \frac{1}{v_1}\left[(x - x_1)^2 + y_1^2\right]^{1/2} + \frac{1}{v_2}\left[(x_2 - x)^2 + y_2^2\right]^{1/2}$$

In order to minimize $T(x)$ we need

$$\frac{dT}{dx} = \frac{1}{v_1}\frac{2(x - x_1)}{\left[(x - x_1)^2 + y_1^2\right]^{1/2}} - \frac{1}{v_2}\frac{2(x_2 - x)}{\left[(x_2 - x)^2 + y_2^2\right]^{1/2}} = 0$$

We recognize the ratios as the sines of the two angles. That is

$$\frac{1}{v_1}\sin\theta_1 - \frac{1}{v_2}\sin\theta_2 \qquad \text{or} \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

In geometric optics, this is known as Snell's Law of Refraction. In that case, the speed of light is given by c/n in a medium with index of refraction n.

Name: ______________________

PHYS2502 Mathematical Physics S23 Quiz #14 27 Apr 2023

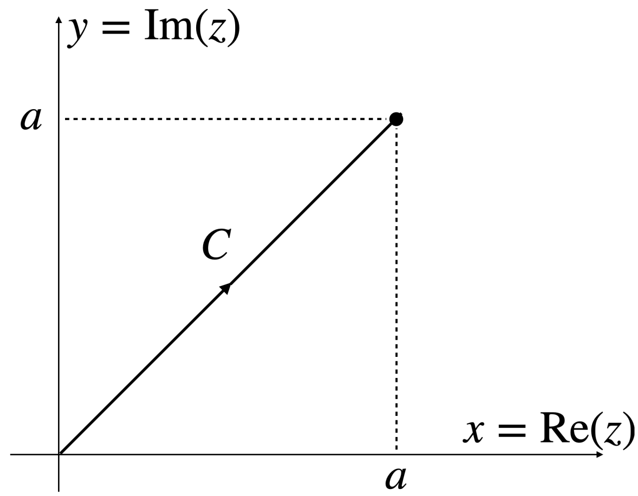
You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

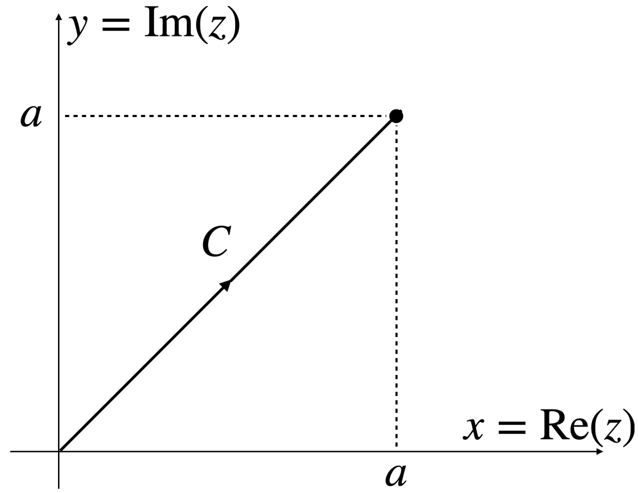
Calculate the integral

$$\mathcal{I} = \int_C f(z)\, dz \qquad \text{where} \qquad f(z) = z^2$$

for the contour C shown on the right.



PHY2502 Mathematical Physics Quiz #14 Solution



Calculate the integral

$$\mathcal{I} = \int_C f(z)\,dz \qquad \text{where} \qquad f(z) = z^2$$

for the contour C shown on the right.

The contour is specified simply as $y = x$ for $0 \le x \le a$, so

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C (x+iy)^2\,(dx + i\,dy) = \int_0^a x^2(1+i)^3\,dx \\
&= (1+i)^3 \int_0^a x^2\,dx = \frac{(1+i)^3}{3}a^3 = \frac{1+3i-3-i}{3}a^3 = -\frac{2}{3}(1-i)a^3
\end{aligned}$$

We can instead be more formal, writing $u = x^2 - y^2$ and $v = 2xy$, and parameterizing the contour as $x = t$ and $y = t$ for $0 \le t \le a$, so, following (8.8)

$$\begin{aligned}
\int_C f(z)\,dz &= \int_0^a \left[(t^2 - t^2)(1) - 2t^2(1)\right]\,dt + i\int_0^a \left[2t^2(1) + (t^2 - t^2)(1)\right]\,dt \\
&= -2(1-i)\int_0^a t^2\,dt = -\frac{2}{3}(1-i)a^3
\end{aligned}$$

It's interesting that we can invoke path independence for analytic functions and just write

$$\int_C f(z)\,dz = \int_0^{a+ia} z^2\,dz = \left.\frac{z^3}{3}\right|_0^{a+ia} = \frac{(1+i)^3}{3}a^3$$

but I'm not 100% certain this isn't an accident.