Name: ______________________

PHYS2502 Mathematical Physics S23 Quiz #14 27 Apr 2023

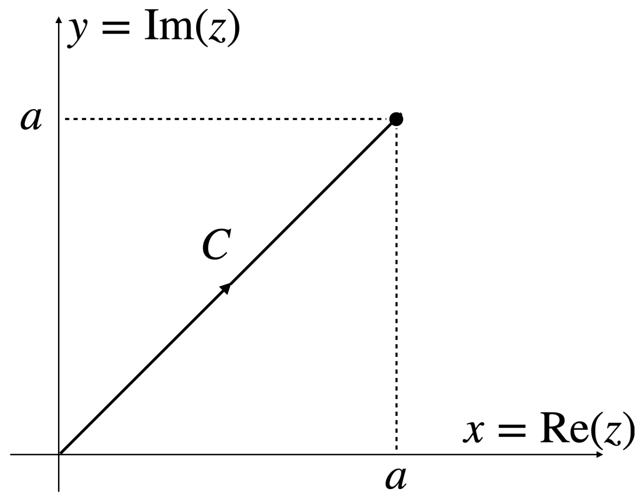
You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Calculate the integral

$$\mathcal{I} = \int_C f(z)\, dz \qquad \text{where} \qquad f(z) = z^2$$

for the contour C shown on the right.



PHY2502 Mathematical Physics Quiz #14 Solution

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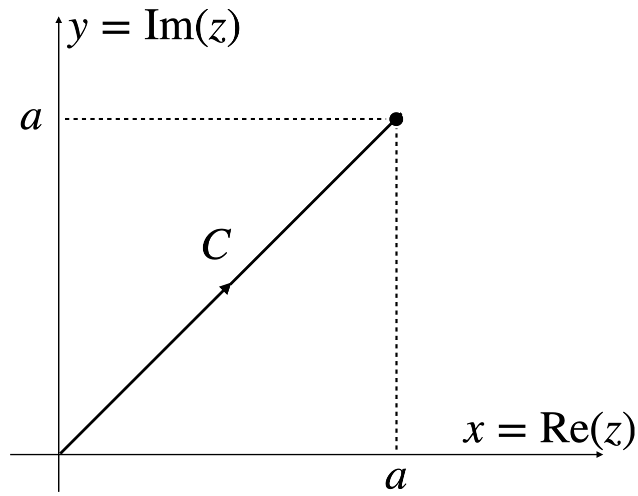

The contour is specified simply as $y = x$ for $0 \leq x \leq a$, so

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (x+iy)^2\,(dx + i\,dy) = \int_0^a x^2(1+i)^3\,dx \\ &= (1+i)^3 \int_0^a x^2\,dx = \frac{(1+i)^3}{3}a^3 = \frac{1+3i-3-i}{3}a^3 = -\frac{2}{3}(1-i)a^3\end{aligned}$$

We can instead be more formal, writing $u = x^2 - y^2$ and $v = 2xy$, and parameterizing the contour as $x = t$ and $y = t$ for $0 \leq t \leq a$, so, following (8.8)

$$\begin{aligned}\int_C f(z)\,dz &= \int_0^a \left[(t^2-t^2)(1) - 2t^2(1)\right]\,dt + i\int_0^a \left[2t^2(1) + (t^2-t^2)(1)\right]\,dt \\ &= -2(1-i)\int_0^a t^2\,dt = -\frac{2}{3}(1-i)a^3\end{aligned}$$

It's interesting that we can invoke path independence for analytic functions and just write

$$\int_C f(z)\,dz = \int_0^{a+ia} z^2\,dz = \left.\frac{z^3}{3}\right|_0^{a+ia} = \frac{(1+i)^3}{3}a^3$$

but I'm not 100% certain this isn't an accident.