Name: ____________________

PHYS2502 Mathematical Physics S23 Quiz #6 23 Feb 2023

You have fifteen minutes to complete this quiz. You may use books, notes, or computers you have with you, but you may not communicate with anyone other than the instructor.

Write your solution on this page, plus the back if necessary, and additional sheets if absolutely necessary. You must show the steps of your solution.

Using an expansion of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, find a recursion relation for the a_n so that this series is a solution to the differential equation

$$y''(x) - xy'(x) - y(x) = 0$$

Show that you get two different solutions which you can argue are independent.

For double credit, show that one of these solutions can be written in a simple, closed analytic form. You can check your answer with direct substitution, if you like.

PHY2502 Mathematical Physics Quiz #6 Solution

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$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
xy'(x) &= x\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \\
0 &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n - a_n\right] x^n = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)a_n\right] x^n \\
a_{n+2} &= \frac{n+1}{(n+2)(n+1)} a_n = \frac{1}{n+2} a_n
\end{aligned}$$

One solution has $a_0 = 1$ and $a_1 = 0$, so just even powers of x, and the other has $a_0 = 0$ and $a_1 = 1$, so just odd powers of x. Just as we argued for the Hermite polynomials, these two power series would be independent of each other.

Now if we consider the even powers and write $n = 2k$ where now $k = 0, 1, 2, \ldots$, then $a_{k+1} = 1/(2k+2)a_k = 1/(k+1)(1/2)a_k$. Setting $a_0 = 1$, we get

$$a_1 = \frac{1}{1}\frac{1}{2}, \qquad a_2 = \frac{1}{2}\frac{1}{2}a_1 = \frac{1}{2!}\frac{1}{2^2}, \qquad \text{so} \qquad a_k = \frac{1}{k!}\frac{1}{2^k}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_k x^{2k} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{2^k} x^{2k} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^2}{2}\right)^k = e^{x^2/2}$$

It is worth checking to make sure $y(x) = e^{x^2/2}$ solves the differential equation.

$$\begin{aligned}
y''(x) - xy'(x) - y(x) &= \frac{d}{dx}(xe^{x^2/2}) - x^2 e^{x^2/2} - e^{x^2/2} \\
&= e^{x^2/2} + x^2 e^{x^2/2} - x^2 e^{x^2/2} - e^{x^2/2} = 0
\end{aligned}$$