PHYS2502 Mathematical Physics Homework #12 Due 18 Apr 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Two horizontal identical circular hoops, each having radius R, are coaxial and separated vertically by a distance $2h$. A continuous soap film is attached to the hoops and drapes between them. Assuming that the surface tension of the film is proportional to its surface area, and that the film is in equilibrium when the surface tension is minimized, find the shape of the soap film. You can ignore the mass of the film, and you can leave your answer in terms of a single undetermined constant.

(2) A point particle of mass m moves in space as a function of time, following the position vector $\vec{r}(t) = \hat{\imath}\, x(t) + \hat{\jmath}\, y(t) + \hat{k}\, z(t)$. Assuming the particle moves according to the Principle of Least Action, that is, the path $\vec{r}(t)$ is the one that minimizes the functional

$$S[\vec{r}(t)] = \int_{t_1}^{t_2} L(x, y, z, \dot{x}, \dot{y}, \dot{z})\, dt \qquad \text{where} \qquad L = \frac{1}{2} m\dot{\vec{r}}^2 - V(\vec{r})$$

show that $\vec{F} = m\vec{a}$, aka "the equation of motion," where $\vec{F} = -\vec{\nabla} V$ and $\vec{a} = \ddot{\vec{r}}$.

(3) Two identical masses m slide on a frictionless horizontal surface and are each connected to a fixed outside wall and to each other by identical springs of stiffness k. The positions of the masses are given by $x_1(t)$ and $x_2(t)$. Knowing that the potential energy of a spring that is compressed or stretched a distance Δ is $k\Delta^2/2$, find the equation of motion for each of the two masses using the Principle of Least Action and the Euler-Lagrange equation. Check your answer against the example we have studied in class.

(4) A particle of mass m moves horizontally on a frictionless surface defined by the x, y plane. Convert to polar coordinates ρ and ϕ, and find the Euler-Lagrange equations of motion. Assuming that potential energy $V = V(\rho)$, that is it has no ϕ-dependence, show that one of the equations of motion leads to a "conserved quantity," that is, something that does not change with time. What is the common name for this conserved quantity?

(5) A highly flexible cable of linear mass density μ and fixed length ℓ hangs motionless in the vertical plane, where its shape minimizes the gravitational potential energy. The cable is fixed at two points at the same vertical position, but separated horizontally by a distance $d < \ell$. Assuming the shape of the cable is given by the function $f(x)$ where x measures the horizontal position, write the integrals that express (a) the gravitational potential energy and (b) the total length of the cable. Combine these integrals and use this to derive a constrained Euler-Lagrange equation that can be solved to find the shape of the hanging cable. Solve this equation for the shape. You don't need to get the result in terms of ℓ and d, but show how you would do that, in principle.