PHYS2502 Mathematical Physics Homework #11 Due 11 Apr 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A matrix $\underline{\underline{A}}$ is unitary if $\tilde{\underline{\underline{A}}} = \underline{\underline{A}}^{-1}$. Prove that the eigenvalues λ of a unitary matrix must be of the form $\lambda = e^{i\phi}$ where ϕ is a real number. (We say that the eigenvalues are "unimodular.") Demonstrate this using the matrix

$$\underline{\underline{A}} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

by showing that it is unitary and then finding its eigenvalues.

(2) Consider two Hermitian matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$. Prove both of the following assertions:

(a) If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ commute, that is if $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$, then the two matrices share a common set of eigenvectors, albeit with (in principle) different eigenvalues. (You can assume that there is a unique set of eigenvectors for any particular Hermitian matrix.)

(b) If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ share a common set of eigenvectors, then they commute. (Remember that any vector can be written as a linear combination of the eigenvectors of any particular Hermitian matrix.)

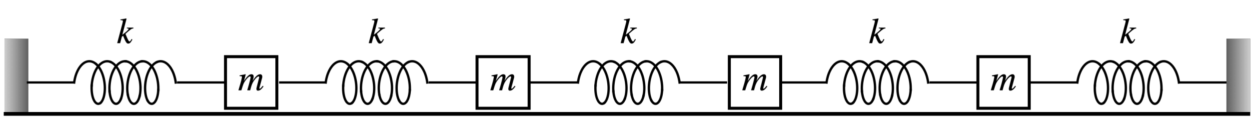
This theorem is critically important for quantum mechanics and the concept of simultaneous measurement.

(3) An "ellipsoid" is a three dimensional surface with three orthogonal symmetry axes, and which appears as an ellipse when viewed along any one axis. Show that the surface described by the points (x, y, z) that satisfy

$$5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4$$

is an ellipsoid. Find the directions of the axes of symmetry. Also, determine the lengths of the symmetry axes. (You are welcome to use MATHEMATICA to help find the eigenvalues and eigenvectors.)

(4) Find the four eigenfrequencies in terms of $\omega_0^2 \equiv k/m$, and describe the amplitudes for the normal modes to which they correspond, for the four masses connected by five springs on a frictionless horizontal surface, as shown below:



(5) For the system shown in Problem (4) above, find and plot the motions of each of the four masses as function of time, when all masses start from rest, with initial positions corresponding to each of the four normal modes. Most of the work for this problem is setting it up correctly in MATHEMATICA, identifying each eigenvector component with the correct mass and frequency.