PHYS2502 Mathematical Physics Homework #8 Due 21 Mar 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Prove the "chain rule" for the divergence operator, namely for a scalar field $f(\vec{r})$ and a vector field $\vec{A}(\vec{r})$,

$$\vec{\nabla}\cdot(f\vec{A}) = \vec{\nabla}f\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A}$$

(2) The time-dependent Schrödinger Equation in three dimensions is

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t) = i\hbar\frac{\partial\psi}{\partial t}$$

where $V(\vec{r})$ is a potential energy function and $\psi(\vec{r},t)$ is called the "wave function." Show that this equation implies that $\rho(\vec{r},t) = \psi^*\psi$ is a conserved density if its current density is given by $j(\vec{r},t) = \hbar\,\mathrm{Im}(\psi^*\vec{\nabla}\psi)/m$.

(3) A magnetic field $\vec{B}(r,\phi) = \hat{\phi}\,B_0\,(r/a)^2\cos^2\phi$ where r and ϕ are the polar coordinates in the (x,y) plane, and B_0 is a constant. Find the total enclosed current passing through a circle of radius a in the (x,y) plane.centered at the origin. Do the necessary line integral directly, and compare to the result you get using Stokes' Theorem.

(4) An electric field $\vec{E}(r,\theta,\phi) = \hat{r}\,E_0\,(r/a)\cos^2\theta$ where r, θ, and and ϕ are the usual spherical coordinates, and E_0 is a constant. Find the total enclosed charge contained in a sphere of radius a centered at the origin. Do the necessary surface integral directly, and compare to the result you get using Gauss' Theorem.

(5) Find the solutions $u(x,y)$ to the partial differential equation

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

separately for each of the following two boundary conditions:

(a) $u(x,y) = 2y+1$ along the line $x=1$

(b) $u(1,1) = 4$, that is, a single point.

You may want to start by looking for a solution $u(x,y) = f(p)$ where $p = p(x,y)$. One way to do this (other than just guessing outright) is to relate the given differential equation to $dp = 0$. That is, the differential equation should be satisfied if $dp = 0$ as x and y change.