PHYS2502 Mathematical Physics Homework #1 Due 24 Jan 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

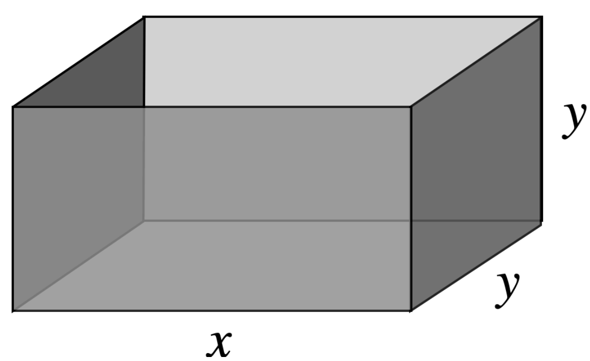
(1) The Hubble constant H_0 measures the expansion rate of the universe. It's value has been measured to be 70 km/sec per Mpc, where one megaparsec (Mpc) is a common measure of cosmological distances, and equals 3.1×10^{19} km. If H_0 is truly a constant in time, then what is the age of the universe? Express your answer in years.

(2) Use dimensional analysis to estimate the energy of an electron with mass m bound in an atom with size $a = 10^{-10}$ m. In this case, the scale is set quantum mechanically according to the quantity $\hbar$, which has the same dimensions as angular momentum. Express your answer in electron volts. (When working in quantum mechanics, it is handy to remember that $\hbar c = 200$ MeV·fm and that $mc^2 = 0.511$ MeV for an electron.)

(3) Use dimensional analysis to find an expression for the pressure at the center of the Sun, assuming it only depends on gravity and the solar mass and radius. Now assume the Sun has uniform density, is made only of hydrogen, and follows the ideal gas law to find the temperature at the center of the Sun.

(4) The equation $ax^2+by^2 = c$, where a, b, and c are positive constants, describes a collection of points (x, y) that lie on an ellipse. Find the two points at which the slope $dy/dx = 1$ in terms of a, b, and c.

(5) You have a fixed number of square feet of lumber with which to build an open box of maximum volume. The box must have square sides, and no top:



Find the ratio of the length of the base to the height of the box.

PHY2502 Mathematical Physics Homework #1 Solutions

(1) The only parameter we have is H_0 which has units of inverse time, so the age of the universe would be $1/H_0 = (1/70)\,3.1 \times 10^{19} = 4.4 \times 10^{17}$ sec = 14 billion years.

(2) Energy has dimensions ML^2T^{-2} and $[\hbar] = ML^2T^{-1}$ so write $E = m^x\hbar^y a^z$ giving

$$ML^2T^{-2} = M^xM^yL^{2y}T^{-y}L^z = M^{x+y}L^{2y+z}T^{-y}$$

so $y = 2$, $z = 2 - 2y = -2$, $x = 1 - y = -1$ and $E = \hbar^2/ma^2 = \hbar^2c^2/(mc^2)a^2$. Therefore $E = (200\ \text{MeV}\cdot\text{fm})^2/(0.511\ \text{MeV}(10^5\ \text{fm})^2) = 7.8\times 10^{-6}$ MeV=7.8 eV, which is about right.

(3) Write $P = G^xM_\odot^yR^z$. Now $[P] = MLT^{-2}/L^2 = ML^{-1}T^{-2}$, so

$$ML^{-1}T^{-2} = M^{-x}L^{3x}T^{-2x}M^yL^z = M^{y-x}L^{3x+z}T^{-2x}$$

and it follows that $x = 1$. Also $y - x = 1$ so $y = 2$, and $3x + z = -1$ so $z = -4$ and we find

$$P = G\frac{M_\odot^2}{R^4} = 6.67 \times 10^{-11}\frac{(1.99 \times 10^{30})^2}{(6.96 \times 10^8)^4} \approx 10^{15}\ \text{N/m}^2$$

Now $PV = nkT = (M_\odot/m_p)kT$ so the temperature at the center is

$$T = P\left(\frac{4}{3}\pi R^3\right)\frac{m_p}{M_\odot}\frac{1}{k} = 10^{15}\frac{4}{3}\pi(6.96 \times 10^8)^3\frac{1.67 \times 10^{-27}}{1.99 \times 10^{30}}\frac{1}{1.38 \times 10^{-23}} \approx 10^8\ \text{K}$$

(4) Differentiating gives $2ax + 2by(dy/dx) = 0$ so the points must satisfy $y = -ax/b$. Substituting into the ellipse equation gives $ax^2 + a^2x^2/b = (a + a^2/b)x^2 = c$. This gives the two solutions for $x = \pm\sqrt{c/(a + a^2/b)}$ with the corresponding values for y.

(5) The volume is $V = xy^2$ and the area is $A = 3xy + 2y^2$. We could solve for x in terms of y and (the constant) A, and then find the derivative of V with respect to y, but I think it is easier to use implicit differentiation, as we did in Problem (4) above. That is

$$\frac{dV}{dx} = y^2 + 2xy\frac{dy}{dx} = 0 \qquad \text{with} \qquad 0 = 3y + 3x\frac{dy}{dx} + 4y\frac{dy}{dx}$$

We want the ratio $r = x/y$, so rewrite the second equation as

$$3 + (3r + 4)\frac{dy}{dx} = 0 \qquad \text{so} \qquad \frac{dy}{dx} = -\frac{3}{3r + 4}$$

and the first equation becomes

$$1 + 2r\left(-\frac{3}{3r + 4}\right) = 0 \qquad \text{or} \qquad 3r + 4 - 6r = 0 \qquad \text{i.e.} \qquad r = \frac{4}{3}$$

PHYS2502 Mathematical Physics Homework #2 Due 31 Jan 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) The motion of a damped harmonic oscillator in one dimension is given by

$$x(t) = Ae^{-\beta t}\cos(\omega t + \phi)$$

Find A and ϕ in terms of the initial conditions $x(0) = x_0$ and $v(0) = v_0$. Assume that A, β, and ω are all real and positive. (You are welcome to solve this in MATHEMATICA, but in this case submit a PDF of your solution notebook.)

(2) Consider a straight rod of length ℓ and mass m. The center of mass of the rod is

$$x_{\mathrm{CM}} = \frac{1}{m}\int_0^L x\, dm$$

where x measures the position along the rod.

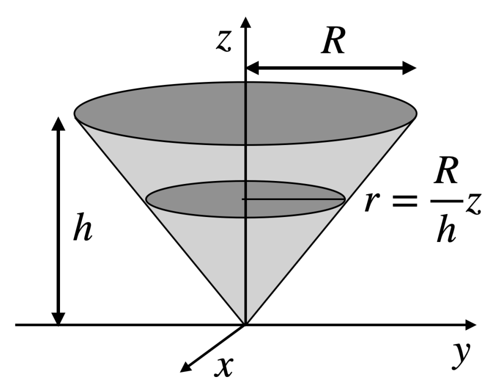
(a) Show that x_{CM} is what you expect if the rod has uniform mass density.

(b) Now calculate x_{CM} assuming that the mass density $\lambda(x)$ of the rod grows linearly from zero at the end of the rod at $x = 0$. Express your answer as a constant times L.

(3) The figure shows an inverted vertical right circular cone of uniform mass density and height h and base radius R, with symmetry around the z-axis. The moment of inertia for an object $\mathcal{O}$ with mass m is given by

$$I = \int_{\mathcal{O}}(x^2 + y^2)\, dm = \int_{\mathcal{O}} \xi^2\, dm$$

where $\xi = (x^2+y^2)^{1/2}$ is the distance from the z-axis for an infinitesimal mass element dm. Find the moment of inertia of the cone in terms of m, h, and R. You might start by finding the moment of inertia of a disk with radius r and thickness dz.



(4) Use the definitions of hyperbolic sine and hyperbolic cosine in terms of exponential functions to prove that

$$\sinh(x+y) = \sinh(x)\cosh(y) + \sinh(y)\cosh(x)$$

(5) Evaluate the following integral

$$\int_0^\infty x^4 e^{-ax^2}\, dx$$

using the techniques described in Section 1.5.6. This integral is used to find the root-mean-square velocity of gas particles that follow the Maxell-Boltzmann Distribution in statistical mechanics.

(1) See the MATHEMATICA notebook. The answers are

$$A = \frac{\sqrt{2\beta v_0 x_0 + v_0^2 + x_0^2\left(\beta^2+\omega^2\right)}}{\omega} \qquad \text{and} \qquad \phi = -\cos^{-1}\left(\frac{x_0\omega}{\sqrt{2\beta v_0 x_0 + v_0^2 + x_0^2\left(\beta^2+\omega^2\right)}}\right)$$

(2) For (a), the mass density is $\lambda = m/L$, so $dm = \lambda dx = (m/L)dx$ and

$$x_{\rm CM} = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\left.\frac{x^2}{2}\right|_0^L = \frac{1}{L}\frac{L^2}{2} = \frac{1}{2}L$$

which is what you expect. For (b), we write $\lambda = ax$ and to determine a we have

$$m = \int_0^L \lambda\,dx = a\int_0^L x\,dx = a\frac{L^2}{2} \qquad \text{so} \qquad a = \frac{2m}{L^2}$$

The center of mass is therefore

$$x_{\rm CM} = \frac{1}{m}\int_0^L ax^2\,dx = \frac{2}{L^2}\left.\frac{x^3}{3}\right|_0^L = \frac{2}{3}L$$

This makes sense. The rod gets heavier as you move to the right, so the center of mass should be somewhat right of center.

(3) Build the cone out of thin disks (which we analyzed in class) of radius $r(z)$ as shown in the figure, each of which has mass $dm = \rho \times \pi r^2\,dz$ and moment of inertia $dm\,r^2/2$. The mass density ρ is the cone mass m divided by its volume V. You can look up the volume of a circular cone, but it's easy enough to calculate, namely

$$V = \int_{\rm O} dV = \int_0^h \pi r^2\,dz = \pi\left(\frac{R}{h}\right)^2\int_0^h z^2\,dz = \pi\frac{R^2}{h^2}\left.\frac{z^3}{3}\right|_0^h = \frac{1}{3}\pi R^2 h$$

Now calculate the moment of inertia by adding up the contributions from the little disks:

$$\begin{aligned} I &= \int_{\rm O}\frac{1}{2}dm\,r^2 = \frac{1}{2}\rho\pi\left(\frac{R}{h}\right)^4\int_0^h z^4\,dz = \frac{1}{2}\rho\pi\left(\frac{R}{h}\right)^4\frac{h^5}{5} \\ &= \frac{1}{2}\frac{m}{\pi R^2 h/3}\pi\frac{R^4 h}{5} = \frac{3}{10}mR^2 \end{aligned}$$

This calculation is also carried out in *Classical Mechanics*, by John Taylor, as Example 10.3, where the distinction is made that our I is the component I_{zz} of the inertia tensor.

(4) I think it is easier to start with the right hand side and show that it equals the left.

$$
\begin{aligned}
& \sinh(x)\cosh(y) + \sinh(y)\cosh(x) \\
= & \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^y - e^{-y}}{2}\frac{e^x + e^{-x}}{2} \\
= & \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}}{4} + \frac{e^{x+y} + e^{-x+y} - e^{x-y} - e^{-x-y}}{4} \\
= & \frac{e^{x+y} - e^{-x-y}}{2} = \sinh(x+y)
\end{aligned}
$$

(5) Start with (1.12), that is

$$
I(a) = \int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\pi} a^{1/2} = \sqrt{\pi} a^{-1/2}
$$

Now take the derivative with respect to a twice.

$$
\begin{aligned}
I'(a) &= \int_{-\infty}^{\infty} (-x^2) e^{-ax^2}\, dx = \sqrt{\pi} a^{1/2} = -\frac{1}{2}\sqrt{\pi} a^{-3/2} \\
I''(a) &= \int_{-\infty}^{\infty} x^4 e^{-ax^2}\, dx = \sqrt{\pi} a^{1/2} = \frac{1}{2}\frac{3}{2}\sqrt{\pi} a^{-5/2}
\end{aligned}
$$

The integral we want is

$$
\int_0^{\infty} x^4 e^{-ax^2}\, dx = \frac{1}{2} I''(1) = \frac{3}{8}\sqrt{\pi}
$$

The answer is easily confirmed with MATHEMATICA.

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PHYS2502 Mathematical Physics Homework #3 Due 7 Feb 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Using an appropriate trigonometric change of variables, show that

$$\tan^{-1}(x) = \int_0^x \frac{du}{1+u^2}$$

and then use this to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(2) Make a plot of the function $\log(1+x)$ over the range $-1 < x \leq 1.5$. Then add to the plot successive approximations by powers of x for $x_0 = 0$. That is, make a figure similar to Fig.2.1 in the Concepts text. I suggest you work this problem out using either MATHEMATICA or your favorite programming language or computer application.

(3) The "effective potential energy" for a planet orbiting a star can be written as

$$U(r) = -\frac{2a}{r} + \frac{b}{r^2}$$

where a and b are positive constants. Find the radius $r = r_0$ that gives the minimum value of $U(r)$, in terms of a and b. (This would be the radius of a circular orbit.) Now you know that the angular frequency for a mass m in a potential energy function $U(x) = kx^2/2$ is $\omega = \sqrt{k/m}$. Use this, and a Taylor expansion of $U(r)$ about the minimum value, to find the approximate period of small oscillations of the planet of mass m about $r = r_0$.

(4) Express $\cos^2(x/2)$ in terms of $\cos x$ by first writing $\cos(x/2)$ using Euler's Formula. Use your result to derive Equation (1.6g) in the Concepts text.

(5) Using Equation (2.16) in the Concepts text, make two plots of the motion $x(t)$ as a function of time t for a simple harmonic oscillator with angular frequency $\omega = 2\pi$ and amplitude $R = 4$. One of the plots should have phase $\phi = +\pi/4$ and the second should have phase $\phi = -\pi/4$. Which of these plots, would you say, "lags" the other by 90°?

(1) If we write $u = \tan y$ (which is the obvious substitution given what we are to show) then

$$1 + u^2 = 1 + \tan^2 y = 1 + \frac{\sin^2 y}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$

Now we also have

$$du = \frac{du}{dy}dy = \frac{d}{dy}\left[\frac{\sin y}{\cos y}\right]dy = \left[\frac{\cos y}{\cos y} - \frac{\sin y}{\cos^2 y}(-\sin y)\right]dy = \left[1 + \frac{\sin^2 y}{\cos^2 y}\right]dy = \frac{dy}{\cos^2 y}$$

Therefore

$$\int_{u=0}^{x} \frac{du}{1+u^2} = \int_{y=0}^{\tan^{-1} x} \cos^2 y \frac{dy}{\cos^2 y} = \int_{y=0}^{\tan^{-1} x} dy = y|_{y=0}^{\tan^{-1} x} = \tan^{-1}(x)$$

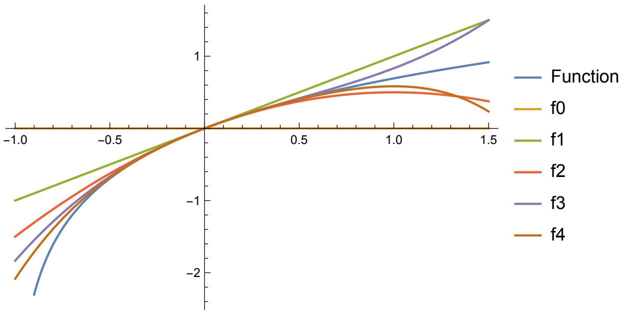
Now $\pi/4 = \tan^{-1}(1)$, so it is more or less obvious that what we want to do is form a Taylor series expansion of the integrand about $u = 0$. We write $f(u) = 1/(1+u^2)$ and then

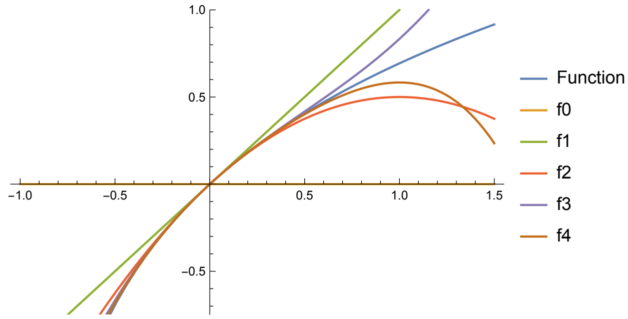
$$\begin{aligned}
& & & f(0) = 1 \\
f'(u) &= \frac{-2u}{(1+u^2)^2} & \text{so} \quad & f'(0) = 0 \\
f^{(2)} &= \frac{6u^2-2}{(u^2+1)^3} & \text{so} \quad & f^{(2)}(0) = -2 \\
f^{(3)} &= -\frac{24u\,(u^2-1)}{(u^2+1)^4} & \text{so} \quad & f^{(3)}(0) = 0 \\
f^{(4)} &= \frac{24\,(5u^4-10u^2+1)}{(u^2+1)^5} & \text{so} \quad & f^{(4)}(0) = 24 \\
f^{(5)} &= -\frac{240u\,(3u^4-10u^2+3)}{(u^2+1)^6} & \text{so} \quad & f^{(5)}(0) = 0 \\
f^{(6)} &= \frac{720\,(7u^6-35u^4+21u^2-1)}{(u^2+1)^7} & \text{so} \quad & f^{(6)}(0) = -720
\end{aligned}$$

(I actually used Mathematica to calculate the derivatives to avoid the tedium.) Then

$$\begin{aligned}
\frac{\pi}{4} = \int_0^1 \frac{du}{1+u^2} &= \int_0^1 \left[1 - \frac{2}{2!}u^2 + \frac{24}{4!}u^4 - \frac{720}{6!}u^6 + \cdots\right] du \\
&= 1 - \frac{2}{2}\frac{1}{3} + \frac{24}{24}\frac{1}{5} - \frac{720}{720}\frac{1}{7} + \cdots = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\end{aligned}$$

(2) See the MATHEMATICA notebook. Here is the plot, with the second version made with an smaller vertical scale.





(3) Finding the radius r_0 at which the potential is a minimum, means that we take the derivative dU/dr and set it equal to zero. So,

$$\frac{dU}{dr} = \frac{2a}{r^2} - 2\frac{b}{r^3} = \frac{2}{r^2}\left[a - \frac{b}{r}\right] = 0 \qquad \text{so} \qquad r_0 = \frac{b}{a}$$

Now if we Taylor expand $U(r)$ about the minimum, the first term is a constant (which does not affect the motion, since the force from it is zero) and the second term, which is proportional to dU/dr at $r = r_0$, is zero by construction. So, the first important term is the quadratic term, which is what gives us the oscillatory motion. That is, we calculate

$$\begin{aligned} \frac{1}{2}\left.\frac{d^2U}{dr^2}\right|_{r=r_0}(r-r_0)^2 &= \frac{1}{2}\left[-\frac{4a}{r^3} + \frac{6b}{r^4}\right]_{r=b/a}(r-r_0)^2 = \frac{1}{2}\left[-\frac{4a^4}{b^3} + \frac{6a^4}{b^3}\right](r-r_0)^2 \\ &= \frac{1}{2}\frac{2a^4}{b^3}(r-r_0)^2 \end{aligned}$$

In other words "k" is $2a^4/b^3$ and the period of small oscillations is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mb^3}{2a^4}}$$

(4) This is pretty simple. We have

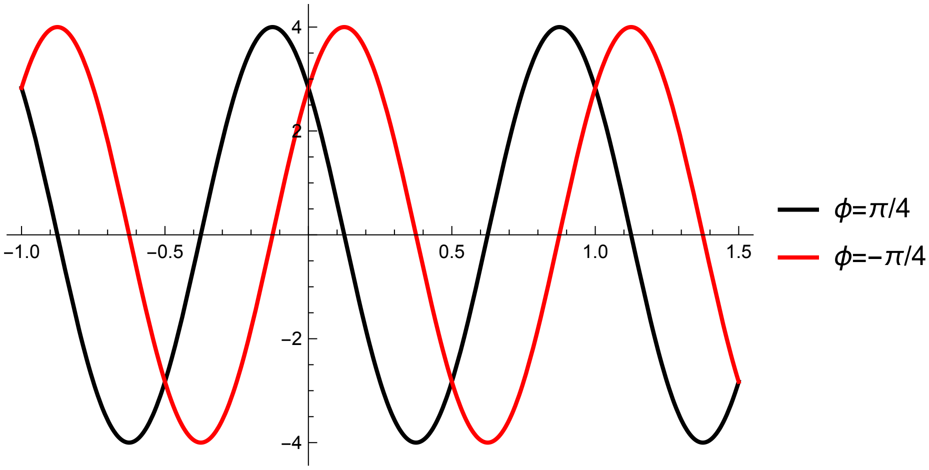
$$\cos^2\left(\frac{x}{2}\right) = \left(\frac{e^{x/2} + e^{-x/2}}{2}\right)^2 = \frac{e^x + 2 + e^{-x}}{4} = \frac{1 + \cos x}{2}$$

which immediately gives you (1.6g).

(5) This is a simple exercise, only trying to make the point about what “lag” means. Equation (2.16) is

$$x(t) = R\cos(\omega t + \phi)$$

The plots are (from the MATHEMATICA notebook)



The plot for $\phi = -\pi/4$ lags behind the plot for $\phi = +\pi/4$, by one quarter of a period, or $\pi/2 = 90°$. The term “phase lag” is often used in electrical engineering or mechanical response, so sometimes the equation is written instead as $x(t) = R\cos(\omega t - \phi)$ so that the phase lag is a positive quantity.

PHYS2502 Mathematical Physics Homework #4 Due 14 Feb 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A function $y = y(x)$ obeys the second order linear differential equation

$$y'' + f(x)y' + g(x)y = 0$$

where $f(x)$ and $g(x)$ are arbitrary functions. Show that if $y_1(x)$ and $y_2(x)$ both solve the differential equation, then the linear combination $y_3 = ay_1(x) + by_2(x)$ also solves the equation, where a and b are arbitrary constants. This is called the *Principle of Superposition.*

(2) Use the "integrating factor" approach to find the solution $y(x)$ to the differential equation

$$\frac{dy}{dx} - y = 2xe^{2x}$$

subject to the boundary condition $y(0) = 1$. Do this by hand, but you are of course welcome to check your answer with MATHEMATICA, either by direct substitution or by using DSolve.

(3) Assume that a spherical raindrop evaporates at a rate proportional to its surface area. If its radius is initially 3 mm, and one hour later its radius is 2 mm, find the radius of the raindrop at any time t.

(4) Given the second order linear differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 1 = 0 \qquad \text{where} \qquad x > 0$$

Find the general solution for the function $y(x)$. (The general solution is a solution with two arbitrary constants.) You can do this by first converting the equation to first order, and then integrating the result. You can check your solution with MATHEMATICA.

(5) Find the unique solution $y(x)$ to the differential equation and boundary conditions

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2x \qquad \text{with} \qquad y(0) = 0 \qquad \text{and} \qquad y'(0) = 1$$

You can do this by using the ansatz $y(x) = e^{\alpha x}$ to find the general solution to the homogeneous equation, and then make a good guess to find a particular solution to the complete equation. Given this, use the boundary conditions to find the two otherwise arbitrary constants from the general solution. You're of course welcome to check your answer with MATHEMATICA.

PHY2502 Mathematical Physics Homework #4 <u>Solutions</u>

(1) Just do the work and the solution falls out:

$$\begin{aligned} y_3'' + f(x)y_3' + g(x)y_3 &= [ay_1'' + by_2''] + f(x)[ay_1' + by_2'] + g(x)[ay_1 + by_2] \\ &= a[y_1'' + f(x)y_1' + g(x)y_1] + b[y_2'' + +f(x)y_2' + g(x)y_2] \\ &= a[0] + b[0] = 0 \end{aligned}$$

(2) This equation has the form (3.6) in the text, namely

$$\frac{dy}{dx} + p(x)y = g(x) \qquad \text{where} \qquad p(x) = -1 \qquad \text{and} \qquad g(x) = 2xe^{2x}$$

In this case, the integrating factor

$$\mu(x) = \exp\left[\int p(x)\,dx\right] = e^{-x}$$

should render the differential equation exact. In fact

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \frac{d}{dx}\left[e^{-x}y\right]$$

and integrating the left hand side becomes simple. Using integration by parts on the right,

$$\int 2xe^x\,dx = 2xe^x - 2\int e^x\,dx = 2(x-1)e^x$$

Therefore, the general solution is

$$e^{-x}y = 2(x-1)e^x + C \qquad \text{or} \qquad y = 2(x-1)e^{2x} + Ce^x$$

Now $y = 1$ when $x = 0$, so

$$1 = -2 + C \qquad \text{so} \qquad C = 3 \qquad \text{and} \qquad y = 2(x-1)e^{2x} + 3e^x$$

It is simple to use MATHEMATICA to confirm that this is the correct answer, either by just substituting this expression into the given differential equation, or by using DSolve to solve the differential equation directly.

(3) The volume of the raindrop is $V = 4\pi r^3/3$ where r is the radius of the raindrop. The surface area of the raindrop is $A = 4\pi r^2$, so we are told that

$$\frac{dV}{dr} = kA$$

where k is a proportionality constant. This reduces to the differential equation

$$4\pi r^2 \frac{dr}{dt} = k4\pi r^2 \qquad \text{or} \qquad \frac{dr}{dt} = k \qquad \text{so} \qquad r(t) = kt + a$$

We can determine k and a from the information given. $r = 3$ mm when $t = 0$, and $r = 2$ mm when $t = 1$ hour, so $a = 3$ mm and

$$2 = k \cdot 1 + 3 \qquad \text{so} \qquad k = -1 \qquad \text{and} \qquad r(t) = 3 - t$$

where the radius is in mm and time is in hours. The raindrop disappears after three hours.

(4) If we write $v = dy/dx$ and do a little rearranging, the equation becomes

$$x^2 \frac{dv}{dx} + 2xv = 1$$

which can be further rearranged and use the integrating factor approach. However, since

$$\frac{d}{dx}(x^2 v) = x^2 \frac{dv}{dx} + 2xv$$

the left side is already exact! This makes the solution simple, namely

$$x^2 v = x + C_1 \qquad \text{so} \qquad v = \frac{dy}{dx} = \frac{x + C_1}{x^2} = \frac{1}{x} + \frac{C_1}{x^2}$$

We get the final solution by integrating again, namely

$$y = \log(x) - \frac{C_1}{x} + C_2$$

It is easy to insert this into the differential equation to confirm that it is a solution, either with MATHEMATICA or by hand.

(5) For the homogeneous solution $y_h(x)$, we need to solve the equation

$$\frac{d^2y_h}{dx^2} + \frac{dy_h}{dx} - 2y_h = 0$$

Using the suggestion $y_h(x) = e^{\alpha x}$ leads to

$$(\alpha^2 + \alpha - 2)e^{\alpha x} = 0 \qquad \text{so} \qquad (\alpha - 1)(\alpha + 2) = 0$$

which means we have two solutions, namely e^x and e^{-2x}. The equation is linear, so we apply superposition and write

$$y_h(x) = C_1 e^x + C_2 e^{-2x}$$

For the particular solution, if we choose $y_p(x) = ax + b$, then we should find values for a and b from the equation, since the second derivative is zero and all that is left is proportional to x. We find

$$0 + a - 2(ax + b) = (a - 2b) - 2ax = 2x$$

in which case $a = -1 = 2b$. Putting this together, the general solution is

$$y(x) = y_h(x) + y_p(x) = C_1 e^x + C_2 e^{-2x} - x - \frac{1}{2}$$

which has the derivative

$$y'(x) = C_1 e^x - 2C_2 e^{-2x} - 1$$

Now $y(0) = 0$ gives $C_1 + C_2 - 1/2 = 0$ and $C_1 - 2C_2 - 1 = 1$, that is

$$C_1 + C_2 = \frac{1}{2} \qquad \text{and} \qquad C_1 - 2C_2 = 2$$

Subtracting the second equation from the first gives $3C_2 = -3/2$ so $C_2 = -1/2$. Then $C_1 = 1/2 - C_2 = 1$ and

$$y(x) = e^x - \frac{1}{2}e^{-2x} - x - \frac{1}{2}$$

Direct substitution with MATHEMATICA confirms that this is the solution.

PHYS2502 Mathematical Physics   Homework #5   Due 21 Feb 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

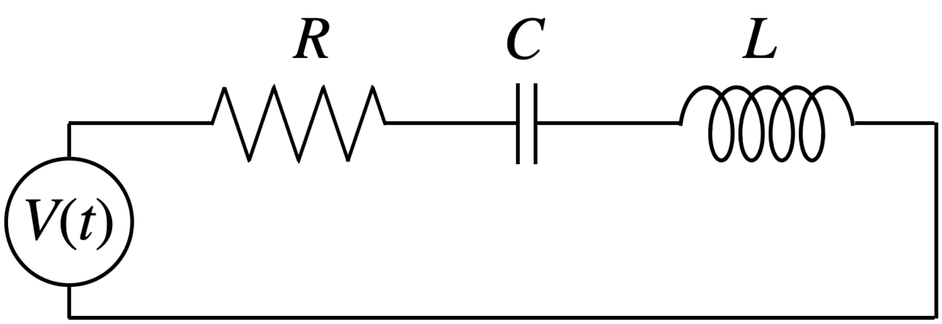
(1) A mass m moves in one dimension $x(t)$ connected to a spring with stiffness k, and is driven by a force term $F = ma\cos\omega t$ where a and ω are constants. Write down and solve the differential equation for $x(t)$ in terms of $\omega_0^2 \equiv k/m$ for the initial conditions $x(0) = \dot{x}(0) = 0$. Use a trigonometric identity to cast your solution in the form of a single product of sines. Write $\omega_0 - \omega = \epsilon$ with $|\epsilon| \ll \omega_0$, and describe the motion for short times $t \ll 1/|\epsilon|$ (but $t \gg 1/\omega_0$) and long times $t \gg 1/|\epsilon|$.

(2) An mechanical oscillator has position $x(t)$ governed by the equations

$$\ddot{x}(t) + 2\dot{x}(t) + 5x(t) = e^{-t}\cos(3t) \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

Find the motion $x(t)$ and plot it for $0 \le t \le 2\pi$. (You can use MATHEMATICA to handle the algebra, if you like, but I want you to solve the differential equation by hand.)

(3) The diagram below is of an electrical circuit with a resistor R, capacitor C, inductor L, and an AC voltage source $V(t)$ connected in series:



The voltage drop across the capacitor is q/C where $q(t)$ is the charge on the capacitor, the voltage drop across the resistor is iR where $i = dq/dt$ is the current in the circuit, and the voltage drop across the inductor is $L\,di/dt$. Kirchoff's Law says that the sum of all voltage drops around a closed path must be zero. If $V(t) = -V_0\cos(\omega t)$, then find $q(t)$ assuming that $q(0) = 0$ and $i(0) = 0$. You are welcome to quote directly from the solution we derived in class for the driven mechanical oscillator.

(4) Find the general solution $y(x)$ for the differential equation $y''(x) = y$ using the series solution approach, about $x = 0$, written as a linear combination of two separate infinite series. Show that that two series are in fact those for $\cosh(x)$ and $\sinh(x)$.

(5) Find a series solution for $y(x)$ about $x = 0$ for the differential equation

$$y'' - 2xy' + \lambda y = 0$$

in terms of two independent series solutions $y_0(x)$ and $y_1(x)$. For what values of λ is the solution a polynomial? Find the polynomial solution for $\lambda = 4$.

(1) The differential equation is given by Newton's Second Law, that is

$$-kx + F = m\ddot{x} \qquad \text{nor} \qquad \frac{d^2x}{dt^2} + \omega_0^2 x = a\cos\omega t$$

Write the solution to the homogeneous equation as $x_h(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$. A more or less obvious form for the particular solution is

$$x_p(t) = A\cos\omega t \qquad \text{so} \qquad -\omega^2 A\cos\omega t + \omega_0^2 A\cos\omega t = a\cos\omega t \qquad \text{and} \qquad A = \frac{a}{\omega_0^2 - \omega^2}$$

Putting these together, the general solution is

$$x(t) = x_h(t) + x_p(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{a}{\omega_0^2-\omega^2}\cos\omega t$$

Now apply the boundary conditions, namely

$$x(0) = c_1 + \frac{a}{\omega_0^2-\omega^2} = 0 \qquad \text{and} \qquad \dot{x}(0) = \omega_0 c_2 = 0$$

which give $c_1 = -a/(\omega_0^2 - \omega^2)$ and $c_2 = 0$, and finally

$$x(t) = \frac{a}{\omega_0^2-\omega^2}\left(\cos\omega t - \cos\omega_0 t\right)$$

Now it's easy to derive that

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

so putting $\alpha - \beta = \omega t$ and $\alpha + \beta = \omega_0 t$ gives $\alpha = (\omega_0+\omega)t/2$ and $\beta = (\omega_0-\omega)t/2$ and

$$x(t) = \frac{2a}{\omega_0^2-\omega^2}\sin\left[\frac{(\omega_0+\omega)t}{2}\right]\sin\left[\frac{(\omega_0-\omega)t}{2}\right] \approx \frac{2a}{\omega_0^2-\omega^2}\sin(\omega_0 t)\sin\left(\frac{\epsilon}{2}t\right)$$

At short times, this oscillates with frequency ω_0 but with an amplitude that grows linear with time. At long times, the oscillations follow a beat pattern.

(2) First tackle the homogenous equation. Using the ansatz $x = e^{\alpha t}$ we find

$$\alpha^2 + 2\alpha + 5 = 0 \qquad \text{so} \qquad \alpha = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

and we can write the homogenous solution as

$$x_h(t) = e^{-t}[c_1\cos(2t) + c_2\sin(2t)]$$

For the particular solution, we can try $x_p(t) = ae^{-t}\cos(3t) + be^{-t}\sin(3t)$ and find values of a and b that give us a solution. We need to calculate

$$\begin{aligned}
\dot{x}(t) &= -ae^{-t}\cos(3t) - 3ae^{-t}\sin(3t) - be^{-t}\sin(3t) + 3be^{-t}\cos(3t) \\
&= -(a-3b)e^{-t}\cos(3t) - (3a+b)e^{-t}\sin(3t) \\
\ddot{x}(t) &= (a-3b)e^{-t}\cos(3t) + (3a-9b)e^{-t}\sin(3t) + (3a+b)e^{-t}\sin(3t) - (9a+3b)e^{-t}\cos(3t) \\
&= (-8a-6b)e^{-t}\cos(3t) + (6a-8b)e^{-t}\sin(3t)
\end{aligned}$$

Now insert this into the differential equation. The factor e^{-t} cancels through and we get

$$(-8a-6b-2a+6b+5a)\cos(3t)+(6a-8b-6a-2b+5b)\sin(3t) = -5a\cos(3t)-5b\sin(3t) = \cos(3t)$$

Therefore $a = -1/5$ and $b = 0$ so the general solution is

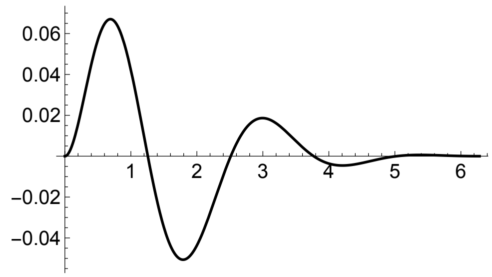
$$x(t) = e^{-t}[c_1\cos(2t) + c_2\sin(2t)] - \frac{1}{5}e^{-t}\cos(3t)$$

See the MATHEMATICA notebook for confirmation that this is the correct solution. Applying the initial conditions, $x(0) = c_1 - 1/5 = 0$ so $c_1 = 1/5$. The derivative is

$$\dot{x}(t) = -e^{-t}[c_1\cos(2t)+c_2\sin(2t)]+e^{-t}[-2c_1\sin(2t)+2c_2\cos(2t)]+\frac{1}{5}e^{-t}\cos(3t)-\frac{1}{5}e^{-t}\sin(3t)$$

so $\dot{x}(0) = -c_1 + 2c_2 + 1/5 = 2c_2 = 0$, so $c_2 = 0$ and the full solution and plot are

$$x(t) = \frac{1}{5}e^{-t}\cos(2t) - \frac{1}{5}e^{-t}\cos(3t)$$



(3) The series LCR circuit is an electric oscillator, in complete analogy with the driven mechanical oscillator. This is easy to see just by applying Kirchoff's Law to the voltage drop in terms of $q(t)$, $i = \dot{q}(t)$, and $di/dt = \ddot{q}(t)$:

$$V(t) + iR + \frac{q}{C} + L\frac{di}{dt} = 0 \qquad \text{so} \qquad \ddot{q}(t) + \frac{R}{L}\dot{q}(t) + \frac{1}{LC}q(t) = \frac{V_0}{L}\cos\omega t$$

Defining $2\beta = R/L$, $\omega_0^2 = 1/LC$, and $\gamma = V_0/L$, we get the same differential equation for $q(t)$ as (3.11) for $x(t)$. All results on the resonant behavior for the circuit follow from here.

This is a nice experiment for the undergraduate instructional laboratory. The amplitude and phase of $q(t)$ can easily be measured as a function of ω by hanging oscilloscope leads across the capacitor.

(4) We proceed in standard fashion. There are no tricks or pitfalls with this particular differential equation.

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \\
y'' - y &= \sum_{n=-}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_n\right]x^n = 0
\end{aligned}$$

Therefore, the recursion relation is $a_{n+2} = a_n/(n+2)(n+1)$. If we choose $a_0 = a$ and $a_1 = 0$, we get

$$a_2 = \frac{1}{2 \cdot 1} a \qquad a_4 = \frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a \qquad \text{so} \qquad a_n = \frac{1}{n!} a$$

for even n. On the other hand, if we take $a_0 = 0$ and $a_1 = b$, then we get

$$a_3 = \frac{1}{3 \cdot 2} b \qquad a_5 = \frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} b \qquad \text{so} \qquad a_n = \frac{1}{n!} b$$

for odd n. The general solution therefore has the form

$$y(x) = a \left[1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots \right] + b \left[x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots \right]$$

Since $d(\cosh x)/dx = \sinh x$ and $d(\sinh x)/dx = \cosh x$, and $\cosh(0) = 1$ and $\sinh(0) = 0$, the Taylor series for these functions look just like the exponential functions but with only the even (cosh) or odd (sinh) terms. That is

$$y(x) = a \cosh x + b \sinh x$$

(5) We start in the standard way and write

$$y(x) = \sum_{n=0}^{\infty} a_0 x^n$$

and substitute into the differential equation. This gives

$$\sum_{n=0}^{\infty} \left\{ n(n-1) a_n x^{n-2} - 2n a_n x^n + \lambda a_n x^n \right\} = \sum_{n=0}^{\infty} \left\{ \left[(n+2)(n+1) a_{n+2} - 2n a_n + \lambda a_n \right] x^n \right\} = 0$$

which gives the recursion relation

$$a_{n+2} = \frac{2n - \lambda}{(n+2)(n+1)} a_n$$

To find $y_0(x)$, put $a_1 = 0$ in which case the only nonzero terms are even powers of x. With $a_0 = 1$ you find

$$y_0(x) = 1 - \frac{1}{2} \lambda x^2 - \frac{\lambda(4-\lambda)}{4 \cdot 3 \cdot 2} x^4 + \cdots$$

For $a_0 = 0$ you only get odd powers, and with $a_1 = 1$ you get

$$y_1(x) = x + \frac{2-\lambda}{3 \cdot 2} x^3 + \frac{(2-\lambda)(6-\lambda)}{5 \cdot 4 \cdot 3 \cdot 2} x^5 + \cdots$$

It is clear that whenever λ is an even integer, for some value of n we will have $\lambda = 2n$ and the series will terminate. Indeed, for $\lambda = 4$, the (finite length) polynomial is the solution $y_0(x)$ with highest power $n = 2$, namely

$$y_0^{(\lambda=4)}(x) = 1 - 2x^2$$

PHYS2502 Mathematical Physics Homework #6 Due 28 Feb 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Starting from the infinite power series expression for the Bessel function $J_m(x)$, where m is an integer, prove that

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x)$$

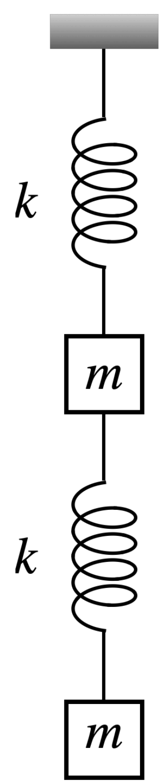
(2) Prove that the Legendre polynomials are "orthogonal", that is $\int_{-1}^{1} P_\ell(x) P_m(x)\, dx = 0$ if $\ell \neq m$. You can do this by writing down the differential equation for $P_\ell(x)$ and multiplying through by $P_m(x)$. Then create a second equation by reversing the indices, subtract the two equations and then integrate.

(3) It is possible to prove that $\int_{-1}^{1} P_\ell(x) P_\ell(x)\, dx = 2/(2\ell+1)$. (But we won't try to do that now.) Use this, along with the orthogonality of Legendre polynomials, to find an expression for the coefficients a_n in the expansion

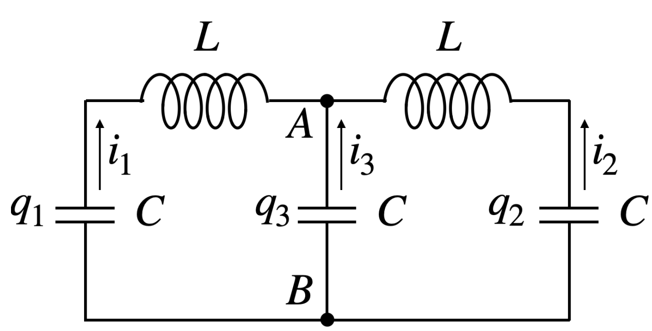
$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x)$$

where $f(x)$ is defined for $-1 \leq x \leq 1$. You can do this by multiplying both sides of this expression by $P_\ell(x)$ and integrating. Now use this find the first few nonzero coefficients for $f(x) = \sin(\pi x)$ and make a plot of the expansion compared to $f(x)$. (Doing the integrals and making the plots is much easier in Mathematica than by hand.)

(4) Two identical mass hang vertically under their own weight from two identical springs from a fixed point on the ceiling, as shown in the figure on the right. Find the two normal frequencies and describe the amplitudes of the two normal modes.



(5) Three identical capacitors C are connected to two identical inductors L as shown in the figure below. Find two coupled differential equations for $q_1(t)$ and $q_2(t)$ and find the normal mode frequencies. Analyze the problem by equating the potential differences for legs 1, 2, and 3 between nodes A and B. Use the sign convention shown for the currents in each of the three legs which implies that $i_1 + i_2 + i_3 = 0$. You can assume the charges are all zero when the currents are all zero.



PHY2502 Mathematical Physics   Homework #6   Solutions

(1) Just follow along from the series expansion for the Bessel function:

$$\begin{aligned}
\frac{d}{dx}\left[x^m J_m(x)\right] &= \frac{d}{dx}\left[x^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{m+2k}\right] \\
&= \frac{d}{dx}\left[2^m \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{m+2k}\right] = \frac{d}{dx}\left[2^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2m+2k}\right] \\
&= 2^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(2m+2k)\left(\frac{x}{2}\right)^{2m+2k-1}\frac{1}{2} \\
&= 2^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k-1)!}\left(\frac{x}{2}\right)^{2m+2k-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m-1+k)!} x^m \left(\frac{x}{2}\right)^{m-1+2k} = x^m J_{m-1}(x)
\end{aligned}$$

(2) Just follow the instructions:

$$\begin{aligned}
(1-x^2)P_m\frac{d^2P_\ell}{dx^2} - 2xP_m\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x)P_m(x) &= 0 \\
(1-x^2)P_\ell\frac{d^2P_m}{dx^2} - 2xP_\ell\frac{dP_m}{dx} + m(m+1)P_\ell(x)P_m &= 0
\end{aligned}$$

Before subtracting these two equations, realize that

$$(1-x^2)P_m\frac{d^2P_\ell}{dx^2} - 2xP_m\frac{dP_\ell}{dx} = \frac{d}{dx}\left\{(1-x^2)P_m\frac{dP_\ell}{dx}\right\} - (1-x^2)\frac{dP_m}{dx}\frac{dP_\ell}{dx}$$

The second term is symmetric between ℓ and m so

$$\frac{d}{dx}\left\{(1-x^2)\left[P_m\frac{dP_\ell}{dx} - P_\ell\frac{dP_m}{dx}\right]\right\} + \left[\ell(\ell+1) - m(m+1)\right]P_\ell(x)P_m(x) = 0$$

When you integrate from -1 to $+1$, the first term is zero because $1-x^2$ is zero at both ends. With $\ell \neq m$, the factor in front of the second term is nonzero, and this leaves you with the orthogonality integral.

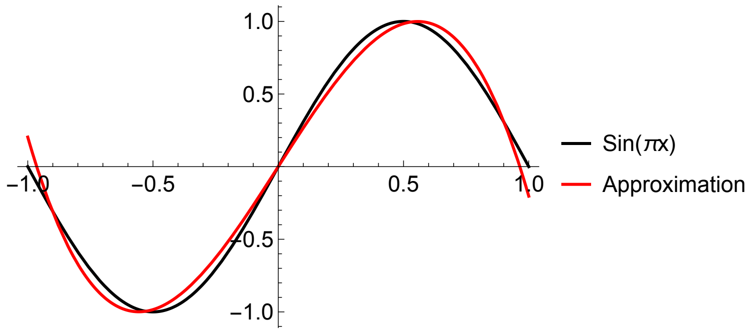
(3) Doing as we are told, we find

$$\int_{-1}^{1} f(x)P_\ell(x)\,dx = \sum_{m=0}^{\infty} a_m \int_{-1}^{1} P_\ell(x)P_m(x)\,dx$$

If $\ell \neq m$, then the relevant terms in the sum on the right side are all zero. If $\ell = m$, however, the only surviving term in the sum is $2a_\ell/(2\ell+1)$. Therefore

$$a_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} f(x)P_\ell(x)\,dx$$

Since $f(x)$ is odd, and the $P_\ell(x)$ are even (odd) for ℓ even (odd), the only nonzero $a)\ell$ are for odd ℓ. The calculation is done in the accompanying MATHEMATICA notebook, and the plot for only terms $\ell = 1, 3$ is the following:



Using more terms makes it hard to see the difference between $f(x)$ and the approximation.

(4) Label the bottom mass #1 and the top mass #2, and let y_1 and y_2 measure the vertical displacements of the masses from their equilibrium positions. (The gravitational force on each mass just moves the equilibrium position downward, so we can ignore it.) The force on mass #1 is $F_1 = -k(y_1 - y_2)$ and the force on mass #2 is $F_2 = -k(y_2 - y_1) - ky_2$, so

$$m\ddot{y}_1 = F_1 = -ky_1 + ky_2 \qquad \text{and} \qquad m\ddot{y}_2 = F_2 = ky_1 - 2ky_2$$

Now define $\omega_0^2 \equiv k/m$ and use $y_1 = a_1 e^{i\omega t}$ and $y_2 = a_2 e^{i\omega t}$ to get

$$(\omega^2 - \omega_0^2)a_1 + \omega_0^2 a_2 = 0 \qquad \text{and} \qquad \omega_0^2 a_1 + (\omega^2 - 2\omega_0^2)a_2 = 0$$

To avoid trivial solutions for a_1 and a_2 we must have

$$(\omega^2 - \omega_0^2)(\omega^2 - 2\omega_0^2) - \omega_0^4 = \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

Solving for ω^2 gives the normal mode frequencies

$$\omega^2 = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2} = \frac{3 \pm \sqrt{5}}{2}\omega_0^2$$

Use the equations above to determine the ratios of the amplitudes in each of the normal modes. For $\omega^2 = [(3+\sqrt{5})/2]\omega_0^2$, we find

$$\frac{a_2}{a_1} = -\frac{\omega^2 - \omega_0^2}{\omega_0^2} = -\frac{3+\sqrt{5}}{2} + 1 = -\frac{\sqrt{5}+1}{2} < 0$$

and the two masses oscillate against each other, with the amplitude of the top mass about 1.6 times the amplitude of the bottom mass. For $\omega^2 = [(3-\sqrt{5})/2]\omega_0^2$, we find

$$\frac{a_2}{a_1} = -\frac{\omega^2 - \omega_0^2}{\omega_0^2} = -\frac{3-\sqrt{5}}{2} + 1 = \frac{\sqrt{5}-1}{2} > 0$$

and the two masses oscillate in phase, with the amplitude of the top mass about 0.6 times the amplitude of the bottom mass.

(5) Integrating $i_1 + i_2 + i_3 = 0$ gives $q_1 + q_2 + q_3 = 0$. Equating the potential differences along the three legs,

$$\frac{q_1}{C} + L\frac{di_1}{dt} = \frac{q_3}{C} = -\frac{q_1}{C} - \frac{q_2}{C} \qquad \text{and} \qquad \frac{q_2}{C} + L\frac{di_2}{dt} = \frac{q_3}{C} = -\frac{q_1}{C} - \frac{q_2}{C}$$

Now use $i_1 = \dot{q}_1$, $i_2 = \dot{q}_2$, divide through by L, and definie $\omega_0^2 = 1/LC$ to get

$$\begin{aligned} \ddot{q}_1 + 2\omega_0^2 q_1 + \omega_0^2 q_2 &= 0 \\ \ddot{q}_2 + \omega_0^2 q_1 + 2\omega_0^2 q_2 &= 0 \end{aligned}$$

These are similar, but not identical, to (3.36). Nevertheless, approach the problem the same way and insert the ansatz $q_1(t) = q_1^0 e^{i\omega t}$ and $q_2(t) = q_2^0 e^{i\omega t}$ to get

$$\begin{aligned} (2\omega_0^2 - \omega^2)q_1^0 + \omega_0^2 q_2^0 &= 0 \\ \omega_0^2 q_1^0 + (2\omega_0^2 - \omega^2)q_2^0 &= 0 \end{aligned}$$

This means we solve for ω in $(2\omega_0^2 - \omega^2)^2 - \omega_0^4 = 0$ for ω, that is

$$\omega^2 - 2\omega_0^2 = \pm\omega_0^2 \qquad \text{so} \qquad \omega^2 = 2\omega_0^2 \pm \omega_0^2 = \omega_0^2, 3\omega_0^2$$

These are the same two normal mode frequencies as for the mechanical oscillator we covered in class.

PHYS2502 Mathematical Physics Homework #7 Due 14 Mar 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Find the equation of a plane which contains the point $(x, y, z) = (1, -2, 5)$ and which is perpendicular to a vector pointing from the origin into the first quadrant and which makes equal angles with the x, y, and z axes. Write your equation in the form $Ax + By + Cz = D$, where A, B, C, and D have numerical values.

(2) A line in space passes through the origin and is at an angle of $45°$ with respect to the positive z-axis and is at equal angles with respect to the positive x- and y-axes. Find the coordinates of the intersection point of this line with the plane in Problem **(1)** above.

(3) For spatial vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$, prove that

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

I think the easiest way to do this is to write the vectors in terms of their components and make use of the Kronecker δ and the Levi-Civita symbol, and their properties.

(4) Use the result from Problem **(3)** to find an expression for $|\vec{A} \times \vec{B}|^2$ in terms of the magnitudes of $\vec{A}$ and $\vec{B}$ and their dot products. Explicitly show that this is the same as the geometric definition of the magnitude of the cross product.

(5) For a particle of mass m moving in a plane located at position $\vec{r}(t)$, find an expression for the kinetic energy

$$K = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2$$

in terms of plane polar coordinates r and ϕ. Do this explicitly by writing $\vec{r}$ first in terms of Cartesian coordinates x and y, convert to polar coordinates, and then take derivatives.

PHY2502 Mathematical Physics Homework #7 Solutions

(1) The normal vector is $\hat{n} = (\hat{x} + \hat{y} + \hat{z})/\sqrt{3}$. For $\vec{r}_0 = \hat{x} - 2\hat{y} + 5\hat{z}$, the plane is

$$(\vec{r} - \vec{r}_0) \cdot \hat{n} = [(x-1) + (y+2) + (z-5)]\,\frac{1}{\sqrt{3}} = 0 \qquad \text{or} \qquad x + y + z = 4$$

(2) Let the unit vector pointing in the direction of the line be $\hat{m} = m_x\hat{x}+m_y\hat{y}+m_z\hat{z}$. Then the equation of the line is $\vec{r} = \hat{m}t$, where t is any real number. Now $m_z = \hat{m}\cdot\hat{z} = \cos\pi/4 = 1\sqrt{2}$, and $m_x = m_y$ where both are positive. Since $m_x^2 + m_y^2 + m_z^2 = 2m_x^2 + 1/2 = 1$, we find that $m_x = 1/2 = m_y$ and the line is $x = t/2$, $y = t/2$, and $z = t/\sqrt{2}$. Inserting into the equation for the plane, $t(1/2 + 1/2 + 1/\sqrt{2}) = t(1 + 1/\sqrt{2})/2 = 4$ so $t = 8\sqrt{2}/(1+\sqrt{2})$ and so

$$\vec{r} = \frac{4\sqrt{2}}{1+\sqrt{2}}\hat{x} + \frac{4\sqrt{2}}{1+\sqrt{2}}\hat{y} + \frac{8}{1+\sqrt{2}}\hat{z}$$

(3) This is pretty straightforward:

$$\begin{aligned}(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) &= \epsilon_{ijk}A_jB_k\epsilon_{imn}C_mD_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})A_jB_kC_mD_n \\ &= A_jB_kC_jD_k - A_jB_kC_kD_j = A_jC_jB_kD_k - A_jD_jB_kC_k \\ &= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})\end{aligned}$$

(4) This is also pretty straightforward:

$$|\vec{A}\times\vec{B}|^2 = (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B}) - (\vec{A}\cdot\vec{B})(\vec{B}\cdot\vec{A}) = A^2B^2 - (AB\cos\psi)^2 = A^2B^2(1-\cos^2\psi) = A^2B^2\sin^2\psi$$

(5) In Cartesian coordinates, $\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$, so $K = m\dot{\vec{r}}^2/2 = m(\dot{x}^2 + \dot{y}^2)/2$. Now

$$\dot{x} = \cos\phi\,\dot{r} - r\sin\phi\,\dot{\phi} \qquad \text{and} \qquad \dot{y} = \sin\phi\,\dot{r} + r\cos\phi\,\dot{\phi}$$

The rest is just algebra:

$$\begin{aligned}K &= \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\left[(\cos\phi\,\dot{r} - r\sin\phi\,\dot{\phi})^2 + (\sin\phi\,\dot{r} + r\cos\phi\,\dot{\phi})^2\right] \\ &= \frac{1}{2}m\left[\cos^2\phi\,\dot{r}^2 - 2r\cos\phi\sin\phi\,\dot{r}\dot{\phi} + r^2\sin^2\phi\,\dot{\phi}^2\right. \\ &\qquad \left. +\sin^2\phi\,\dot{r}^2 + 2r\sin\phi\cos\phi\,\dot{r}\dot{\phi} + r^2\cos^2\phi\,\dot{\phi}^2\right] \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2)\end{aligned}$$

PHYS2502 Mathematical Physics Homework #8 Due 21 Mar 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Prove the "chain rule" for the divergence operator, namely for a scalar field $f(\vec{r})$ and a vector field $\vec{A}(\vec{r})$,

$$\vec{\nabla} \cdot (f\vec{A}) = \vec{\nabla} f \cdot \vec{A} + f\,\vec{\nabla} \cdot \vec{A}$$

(2) The time-dependent Schrödinger Equation in three dimensions is

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{r},t) + V(\vec{r})\psi(\vec{r},t) = i\hbar\frac{\partial\psi}{\partial t}$$

where $V(\vec{r})$ is a potential energy function and $\psi(\vec{r},t)$ is called the "wave function." Show that this equation implies that $\rho(\vec{r},t) = \psi^*\psi$ is a conserved density if its current density is given by $j(\vec{r},t) = \hbar\,\mathrm{Im}(\psi^*\vec{\nabla}\psi)/m$.

(3) A magnetic field $\vec{B}(r,\phi) = \hat{\phi}\,B_0\,(r/a)^2\cos^2\phi$ where r and ϕ are the polar coordinates in the (x,y) plane, and B_0 is a constant. Find the total enclosed current passing through a circle of radius a in the (x,y) plane.centered at the origin. Do the necessary line integral directly, and compare to the result you get using Stokes' Theorem.

(4) An electric field $\vec{E}(r,\theta,\phi) = \hat{r}\,E_0\,(r/a)\cos^2\theta$ where r, θ, and and ϕ are the usual spherical coordinates, and E_0 is a constant. Find the total enclosed charge contained in a sphere of radius a centered at the origin. Do the necessary surface integral directly, and compare to the result you get using Gauss' Theorem.

(5) Find the solutions $u(x,y)$ to the partial differential equation

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

separately for each of the following two boundary conditions:

(a) $u(x,y) = 2y+1$ along the line $x = 1$

(b) $u(1,1) = 4$, that is, a single point.

You may want to start by looking for a solution $u(x,y) = f(p)$ where $p = p(x,y)$. One way to do this (other than just guessing outright) is to relate the given differential equation to $dp = 0$. That is, the differential equation should be satisfied if $dp = 0$ as x and y change.

(1) It is easiest to do this with Cartesian coordinates, making use of the summation notation:

$$\vec{\nabla}\cdot(f\vec{A}) = \frac{\partial}{\partial x_i}(f\,A_i) = \frac{\partial f}{\partial x_i}A_i + f\frac{\partial A_i}{\partial x_i} = \vec{\nabla}f\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A}$$

(2) Multiply the Schrödinger Equation through by ψ^*. Next, take the complex conjugate. of the Schrödinger Equation and multiply through by ψ:

$$\begin{aligned} -\frac{\hbar^2}{2m}\psi^*\vec{\nabla}^2\psi + V(\vec{r})\psi^*\psi &= i\hbar\psi^*\frac{\partial\psi}{\partial t} \\ -\frac{\hbar^2}{2m}\psi\vec{\nabla}^2\psi^* + V(\vec{r})\psi^*\psi &= -i\hbar\psi\frac{\partial\psi^*}{\partial t} \end{aligned}$$

Now subtract these two equations. Note that the potential energy term cancels due to the fact that $V(\vec{r})$ is a real number. You get

$$-\frac{\hbar^2}{2m}\left[\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*\right] = i\hbar\left[\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right] = i\hbar\frac{\partial}{\partial t}(\psi^*\psi) = i\hbar\frac{\partial\rho}{\partial t}$$

For the left side, make use of Problem (1) to get

$$\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) = \vec{\nabla}\psi^*\cdot\vec{\nabla}\psi + \psi^*\vec{\nabla}^2\psi - \vec{\nabla}\psi\cdot\vec{\nabla}\psi^* - \psi\vec{\nabla}^2\psi^* = \psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*$$

The difference between complex conjugates is just twice the imaginary part times i, so

$$\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^* = 2i\,\mathrm{Im}(\psi^*\vec{\nabla}\psi)$$

Our subtracted Schrödinger Equations then become

$$-\frac{\hbar^2}{2m}\vec{\nabla}\cdot\left[2i\,\mathrm{Im}(\psi^*\vec{\nabla}\psi)\right] = i\hbar\frac{\partial\rho}{\partial t} \qquad \text{or} \qquad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \qquad \text{where} \qquad \vec{j} = \frac{\hbar}{m}\mathrm{Im}(\psi^*\vec{\nabla}\psi)$$

(3) The total enclosed current comes from Ampere's Law, namely (in CGS units)

$$I_{\text{enclosed}} = \frac{c}{4\pi}\oint_C \vec{B}\cdot d\vec{\ell}$$

In this case, with C being a circle of radius $r = a$ centered at the origin, it is easiest to just evaluate the line integral directly. Since we have $d\vec{\ell} = \hat{\phi}a\,d\phi$, we write

$$\oint_C \vec{B}\cdot d\vec{\ell} = \int_0^{2\pi} B_0(1)\cos^2\phi\,a\,d\phi = \frac{1}{2}B_0 a\int_0^{2\pi}(1+\cos 2\phi)\,d\phi = \pi B_0 a$$

which gives $I_{\text{enclosed}} = cB_0a/4$, We can also do the integral with Stokes' Theorem, that is

$$\begin{aligned} \oint_C \vec{B}\cdot d\vec{\ell} &= \int(\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \int_S\left(\hat{k}\frac{1}{r}\frac{\partial[rB_\phi]}{\partial r}\right)\cdot\hat{k}\,dS \\ &= \int_0^a\int_0^{2\pi} r\,dr\,d\phi\frac{1}{r}\frac{\partial}{\partial r}\left[rB_0\left(\frac{r}{a}\right)^2\cos^2\phi\right] = 3\pi B_0\frac{1}{a^2}\int_0^a r^2\,dr = \pi B_0 a \end{aligned}$$

(4) The total enclosed charge Q is from Gauss' Law, and the surface integral can be done directly. The electric field has only a $\hat{r}$ component and $d\vec{S} = \hat{r}\, dS$, so

$$\begin{aligned} Q_{\text{enclosed}} &= \frac{1}{4\pi}\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{4\pi}\oint_S E_0\frac{a}{a}\cos^2\theta\, dS = \frac{E_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\, a^2\sin\theta\, d\theta\, d\phi \\ &= \frac{E_0}{4\pi}a^2\, 2\pi\int_{-1}^{1}\mu^2\, d\mu = \frac{1}{3}E_0 a^2 \end{aligned}$$

where I made the substitutions $\mu = \cos\theta$ with $d\mu = -\sin\theta\, d\theta$. To do the integral with Gauss' Theorem, we have

$$\begin{aligned} Q_{\text{enclosed}} &= \frac{1}{4\pi}\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{4\pi}\int_V \vec{\nabla}\cdot\vec{E}\, dV = \frac{1}{4\pi}\int_V \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 E_0\frac{r}{a}\cos^2\theta\right] \\ &= \frac{E_0}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\int_0^a r^2\, dr\frac{1}{r^2}3r^2\frac{1}{a}\cos^2\theta \\ &= \frac{E_0}{4\pi}2\pi\int_{-1}^{1}\mu^2\, d\mu\int_0^a r^2\, dr\frac{3}{a} = \frac{E_0}{2}\frac{3}{a}\frac{2}{3}\frac{a^3}{3} = \frac{1}{3}E_0 a^2 \end{aligned}$$

(5) We start by realizing that

$$\frac{\partial u}{\partial x} = \frac{df}{dp}\frac{\partial p}{\partial x} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{df}{dp}\frac{\partial p}{\partial y}$$

In this case, our differential equation becomes

$$x\frac{df}{dp}\frac{\partial p}{\partial x} - 2y\frac{\partial u}{\partial y} = \frac{df}{dp}\frac{\partial p}{\partial y} = \left[x\frac{\partial p}{\partial x} - 2y\frac{\partial p}{\partial y}\right]\frac{df}{dp} = 0$$

If we set the expression in square brackets to be zero, and compare this to

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$$

we see that the two expression become the same if

$$\frac{\partial p/\partial x}{\partial p/\partial y} = \frac{2y}{x} = -\frac{dy}{dx} \qquad \text{or} \qquad \frac{2}{x}\, dx = -\frac{1}{y}\, dy$$

Integrating this gives $2\log x = -\log y - \log c$ or $\log(cx^2y) = 0$ where c is a constant. In other words, if $p = x^2y$ remains constant as x and y vary, then $u(x,y) = f(p) = f(x^2y)$ is a solution to the original differential equation. (You actually might have gotten there by guessing from inspection.)

Now for boundary condition (a), we need $f(x^2y) = 2y + 1$ when $x = 1$, so

$$u(x,y) = 2(x^2y) + 1 = 2x^2y + 1$$

Boundary condition (b) is defined at just one point, and not along a line, so the answer might be more flexible. Indeed, $u(x,y) = 4x^2y$ is one solution, but so is $u(x,y) = 3x^2y + 1$. For that matter, $u(x,y) = 4$ also works. In fact, any solution of the form $u(x,y) = 4 + g(x^2y)$, where $g(p)$ is only constrained by $g(1) = 0$, fills the bill.

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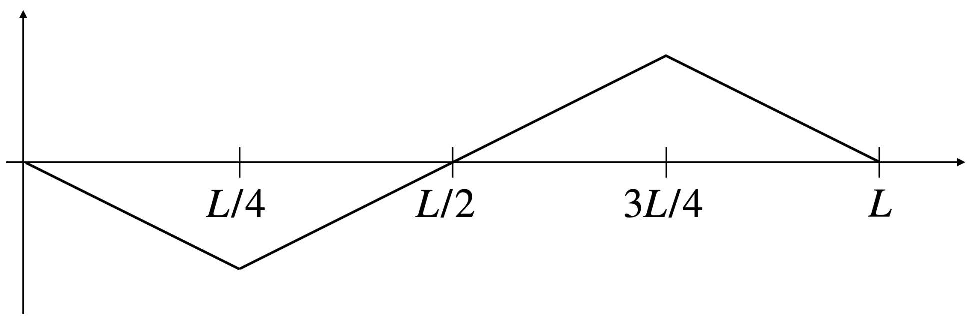
PHYS2502 Mathematical Physics Homework #9 Due 28 Mar 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A string with fixed ends at $x = 0$ and $x = L$ has a mass density μ per unit length and is under tension T. Find the vertical motion $u(x, t)$ of the shape of the string if it is initially flat, that is $u(x, 0) = 0$, but has an initial vertical velocity profile $\dot{u}(x, 0) = V \sin(3\pi x/L)$.

(2) A string with fixed ends at $x = 0$ and $x = L$ has an initial shape give by



Use a Fourier decomposition to find the shape of the string as a function of time, using enough terms in the expansion so that the true shape is clear. Plot the shape at various times within one fundamental period (or make an animation).

(3) A string with fixed ends at $x = 0$ and $x = L$ has a total mass M and is under tension T. Assume the string is vibrating in normal mode n. Integrate over the length of the string to find its total kinetic energy as a function of time. You can assume the solution for the string motion where it is initially at rest, and express your result in terms of the Fourier coefficients B_n in (5.12).

(4) Find the Fourier Transform $A(k)$ of a pulse $f(x) = C(x^2 - a^2)^2$, where C is a constant, for $-a \leq x \leq a$ and $f(x) = 0$ for $|x| > a$. Plot $f(x)$ and $A(k)$ and briefly compare them. Find the RMS width of $f(x)$ and of $A(k)$, and show that their product is independent of a.

(5) Use the generalized definition of the δ-function to show that

$$\delta(x) = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

(1) The general approach using separation of variables gives

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \left[C_n \cos\left(\frac{n\pi vt}{L}\right) + D_n \sin\left(\frac{n\pi vt}{L}\right)\right]$$

which satisfies $u(0,t) = 0 = u(L,t)$ by design. The "flat" initial shape means that

$$u(x,0) = \sum_{n=1}^{\infty} C_n B_n \sin\left(\frac{n\pi x}{L}\right) = 0$$

which implies that $C_n = 0$ for all n. Defining $A_n = B_n D_n$ we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi vt}{L}\right)$$

The vertical velocity of the string is therefore

$$\dot{u}(x,t) = \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi v}{L} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right)$$

The initial vertical velocity of the string is

$$\dot{u}(x,0) = \sum_{n=1}^{\infty} \frac{n\pi v}{L} A_n \sin\left(\frac{n\pi x}{L}\right) = V \sin\left(\frac{3\pi x}{L}\right)$$

We don't need to do any fancy Fourier analysis to find the A_n in this case, because it is obvious that $A_n = 0$ except for $n = 3$, in which case $A_n = LV/3\pi v$. Therefore the complete solution is

$$u(x,t) = \frac{LV}{3\pi v} \sin\left(\frac{3\pi x}{L}\right) \sin\left(\frac{3\pi vt}{L}\right)$$

(2) See the accompanying Mathematica notebook.

(3) This problem is, more or less, an illustration of Parseval's Theorem.

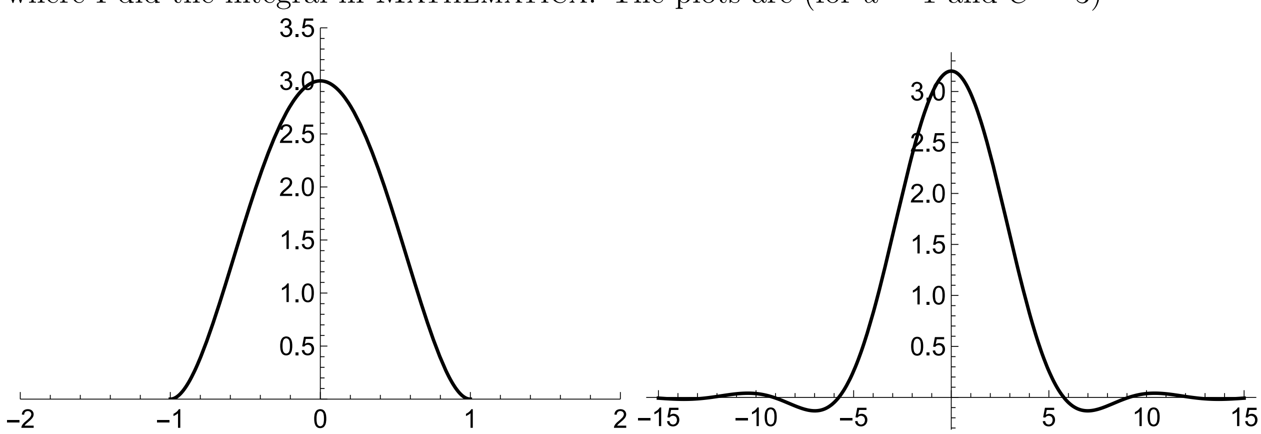
$$
\begin{aligned}
K &= \int_0^L \frac{1}{2}\mu\, dx \left(\frac{\partial u}{\partial t}\right)^2 \\
&= \int_0^L \frac{1}{2}\mu\, dx \left[\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi v}{L}\sin\left(\frac{n\pi v t}{L}\right)\right]\left[\sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi x}{L}\right)\frac{n\pi v}{L}\sin\left(\frac{m\pi v t}{L}\right)\right] \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{2}B_n B_m\,\mu\left(\frac{\pi v}{L}\right)^2 mn\left[\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx\right]\sin\left(\frac{n\pi v t}{L}\right)\sin\left(\frac{m\pi v t}{L}\right) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{2}B_n B_m\,\mu\left(\frac{\pi v}{L}\right)^2 mn\left[\frac{L}{2}\delta_{mn}\right]\sin\left(\frac{n\pi v t}{L}\right)\sin\left(\frac{m\pi v t}{L}\right) \\
&= \frac{1}{4}\sum_{n=1}^{\infty} B_n^2 M\left(\frac{n\pi v}{L}\right)^2\sin^2\left(\frac{n\pi v t}{L}\right)
\end{aligned}
$$

where $M = \mu L$ is the mass of the string. This agrees, essentially, with the result given in Section 5.13 of *Introduction to Vibrations and Waves* by Rankin and Pain (2015).

(4) The Fourier Transform is (5.18), and $f(-x) = f(x)$, so

$$
\begin{aligned}
A(k) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx = \int_{-\infty}^{\infty}[\cos(kx) - i\sin(kx)]f(x)\,dx = \int_{-\infty}^{\infty}\cos(kx)f(x)\,dx \\
&= \int_{-a}^{a} C(x^2-a^2)^2\cos(kx)\,dx = \frac{16}{k^5}\left[\left(3-a^2k^2\right)\sin(ak) - 3ak\cos(ak)\right]
\end{aligned}
$$

where I did the integral in MATHEMATICA. The plots are (for $a = 1$ and $C = 3$)



The width of the functions come from (5.20). We find

$$
\Delta x = \frac{a}{\sqrt{7}} \qquad \text{and} \qquad \Delta k = \frac{2}{a} \qquad \text{so} \qquad \Delta x \Delta k = \frac{2}{\sqrt{7}} = 0.756 > \frac{1}{2}
$$

The widths are consistent with the plots, and the product is consistent with the "uncertainty principle."

(5) From (1.13), we see that for all values of n,

$$\int_{-\infty}^{\infty} e^{-n^2x^2}\,dx = \sqrt{\frac{\pi}{n^2}} = \frac{\sqrt{\pi}}{n} \qquad \text{so} \qquad \int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}\,dx = 1$$

so this function maintains unit area as $n \to \infty$. Now for all finite values of x, the function goes to zero as $n \to \infty$ because the exponential falls more rapidly than n grows. However, for $x = 0$, the function is infinite. This is all we need to call it a δ-function.

PHYS2502 Mathematical Physics Homework #10 Due 4 Apr 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Write out the three equations for x, y, and z represented by

$$\underline{\underline{A}}\,\underline{X} = \underline{C} \qquad \text{where} \qquad \underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \underline{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and solve them for x, y, and z. Then determine the matrix $\underline{\underline{A}}^{-1}$ by writing your answer as $\underline{X} = \underline{\underline{A}}^{-1}\underline{C}$. You might want to check your answer using MATHEMATICA.

(2) Find the inverse matrix $\underline{\underline{A}}^{-1}$ for the matrix $\underline{\underline{A}}$ in Problem (1) by calculating the determinant using an expansion in minors, and then forming the matrix of cofactors.

(3) Construct a 3×3 matrix that rotates a three-dimensional vector $\underline{v}$ through an angle θ about the x-axis, combined with a reflection about the yz-plane, that is, takes x to $-x$. Pick two specific examples for $\underline{v}$ and show that your matrix does what it is supposed to do.

(4) Prove that if $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{0}}$ for two matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$, then the determinant of at least one of them must be zero. Find an example, however, of two 3×3 matrices that are each nonzero, and in fact do not have any full rows or columns with all zeros, but whose product $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{0}}$.

(5) A Lorentz transformation tells you how to convert space and time between two reference frames, call them the "primed" and "unprimed" frames, moving at a velocity v relative to each other, in accordance with the framework of Special Relativity. For a reference frame moving in the x-direction with respect to another frame, the Lorentz transformation is

$$x' = \gamma(x - vt) \qquad \text{and} \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c$, with c being the speed of light. Then define a vector

$$\underline{x} = \begin{bmatrix} ct \\ x \end{bmatrix}$$

(a) Show that the transformation maintains the value of $s^2 = (ct)^2 - x^2$. (We say that the Lorentz transformation maintains the norm of a vector with a "Minkowski metric," instead of a "Euclidean metric.")

(b) Find the Lorentz transformation matrix $\underline{\underline{\Lambda}}$ which takes you from the unprimed frame to the primed frame, by acting on $\underline{x}$.

(c) Write $\underline{\underline{\Lambda}}$ in terms of a single parameter η which combines γ and β. Compare this to a rotation matrix in two dimensions.

(d) Show that the inverse transformation $\underline{\underline{\Lambda}}^{-1}$ corresponds to $v \to -v$ or, equivalently, to the change η to $-\eta$.

PHY2502 Mathematical Physics   Homework #10   Solutions

(1) The equations are

$$\begin{aligned} x + 2y + 4z &= c_1 \\ 2x + 0y + z &= c_2 \\ x + y + z &= c_3 \end{aligned}$$

The second equation says that $z = c_2 - 2x$ and if you subtract the third equation from the first, you get $y + 3z = c_1 - c_3$, so $y = c_1 - c_3 - 3z = c_1 - 3c_2 - c_3 + 6x$. Therefore

$$x + y + z = x + c_1 - 3c_2 - c_3 + 6x + c_2 - 2x = c_3 \qquad \text{so} \qquad x = \frac{1}{5}[-c_1 + 2c_2 + 2c_3]$$

It follows that

$$\begin{aligned} y &= c_1 - 3c_2 - c_3 + \frac{6}{5}[-c_1 + 2c_2 + 2c_3] = \frac{1}{5}[-c_1 - 3c_2 + 7c_3] \\ \text{and} \qquad z &= c_2 - \frac{2}{5}[-c_1 + 2c_2 + 2c_3] = \frac{1}{5}[2c_1 + c_2 - 4c_3] \end{aligned}$$

which can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 & 2 & 2 \\ -1 & -3 & 7 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

which agrees with what MATHEMATICA tells me. I had to check, though, it's easy to make algebra errors and I made a few myself.

(2) First find the determinant by using minors across the first row:

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 1(-1) - 2(1) + 4(2) = 5$$

Now we have to find the cofactor matrix $\underline{\underline{C}}$, where C_{ij} equals $(-1)^{i+j}$ times the determinant of the matrix left when we cross off row i and column j of $\underline{\underline{A}}$. Do these one-by-one:

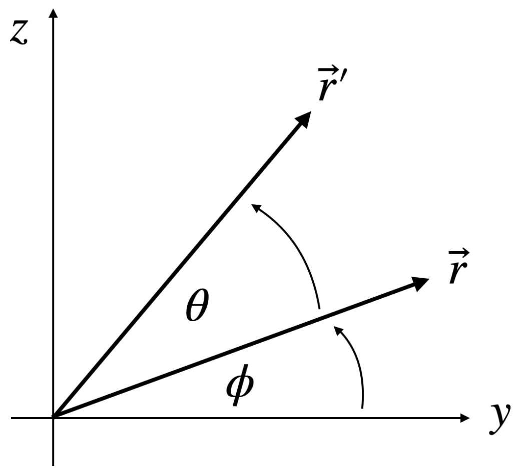
$$C_{11} = (+)\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \qquad C_{12} = (-)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1 \qquad C_{13} = (+)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = +2$$

$$C_{21} = (-)\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = +2 \qquad C_{22} = (+)\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3 \qquad C_{23} = (-)\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = +1$$

$$C_{31} = (+)\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = +2 \qquad C_{32} = (-)\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = +7 \qquad C_{33} = (+)\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

You can see at a glance that $\underline{\underline{A}}^{-1} = \underline{\underline{C}}^{\mathsf{T}}/|\underline{\underline{A}}|$ by comparing with the result from Problem (1).

(3) For a counter clockwise rotation about the x-axis, we have to following to maintain a right-handed coordinate system:



So, following what we did in class but with $x \to y$ and $y \to z$, and also including a reflection $x \to -x$, we have

$$\underline{\underline{D}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Consider a vector $\vec{v} = \hat{i}v_x + \hat{j}v_y$ in the xy plane, and do a 90° rotation. You get

$$\underline{\underline{D}v} = \underline{\underline{D}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} -v_x \\ 0 \\ v_y \end{bmatrix}$$

This is just what you expect. The x-component reverses sign, and there is now a z-component with the same value as the original y-component.

(4) Since $|\underline{\underline{A}}\,\underline{\underline{B}}| = |\underline{\underline{A}}|\,|\underline{\underline{B}}|$ and $|\underline{\underline{0}}| = 0$, $|\underline{\underline{A}}|\,|\underline{\underline{B}}| = 0$ and of the determinants must be zero. This can happen, of course, if matrix itself is nonzero. For example, the determinant of

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is clearly zero. If I construct a matrix of the form

$$\underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ -b_{11} & -b_{12} & -b_{13} \\ b_{11} & b_{12} & b_{13} \end{bmatrix}$$

Then the product will be zero, although $\underline{\underline{B}}$ also has zero determinant. I haven't tried very hard, but I wonder if it is easy to find a matrix that gives a zero product but is not singular.

(5) The first part is straightforward. Just do the algebra.

$$\begin{aligned} s'^2 &= (ct')^2 - x'^2 = c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2 - \gamma^2(x-vt)^2 \\ &= c^2\gamma^2t^2 - 2\gamma^2 tvx + \gamma^2v^2x^2/c^2 - \gamma^2x^2 + 2\gamma^2xvt - \gamma^2v^2t^2 \\ &= c^2\gamma^2t^2\left(1 - \frac{v^2}{c^2}\right) - \gamma^2x^2\left(1 - \frac{v^2}{c^2}\right) = (ct)^2 - x^2 = s^2 \end{aligned}$$

The matrix $\underline{\underline{\Lambda}}$ is simple to construct just by looking at the transformation equations:

$$\begin{aligned} ct' = \gamma ct - \gamma\frac{v}{c}x &= \gamma ct - \gamma\beta x \\ x' = -\gamma\frac{v}{c}ct + \gamma x &= -\gamma\beta ct + \gamma x \\ \text{so} \qquad \underline{\underline{\Lambda}} &= \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \end{aligned}$$

Observing that $\gamma^2 - (\gamma\beta)^2 = \gamma^2(1-\beta^2) = 1$, it is possible to write $\gamma = \cosh\eta$ and $\gamma\beta = \sinh\eta$. In other words, the Lorentz transformation matrix becomes

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix}$$

The similarity to a rotation in two dimensions is striking. Since $v \to -v$ means $\beta \to -\beta$ means $\gamma\beta \to -\gamma\beta$ means $\eta \to -\eta$, we have

$$\begin{aligned} \underline{\underline{\Lambda\Lambda^{-1}}} &= \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix}\begin{bmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{bmatrix} \\ &= \begin{bmatrix} \cosh^2\eta - \sinh^2\eta & \cosh\eta\sinh\eta - \cosh\eta\sinh\eta \\ -\cosh\eta\sinh\eta + \cosh\eta\sinh\eta & -\sinh^2\eta + \cosh^2\eta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

PHYS2502 Mathematical Physics Homework #11 Due 11 Apr 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) A matrix $\underline{\underline{A}}$ is unitary if $\tilde{\underline{\underline{A}}} = \underline{\underline{A}}^{-1}$. Prove that the eigenvalues λ of a unitary matrix must be of the form $\lambda = e^{i\phi}$ where ϕ is a real number. (We say that the eigenvalues are "unimodular.") Demonstrate this using the matrix

$$\underline{\underline{A}} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

by showing that it is unitary and then finding its eigenvalues.

(2) Consider two Hermitian matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$. Prove both of the following assertions:

(a) If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ commute, that is if $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$, then the two matrices share a common set of eigenvectors, albeit with (in principle) different eigenvalues. (You can assume that there is a unique set of eigenvectors for any particular Hermitian matrix.)

(b) If $\underline{\underline{A}}$ and $\underline{\underline{B}}$ share a common set of eigenvectors, then they commute. (Remember that any vector can be written as a linear combination of the eigenvectors of any particular Hermitian matrix.)

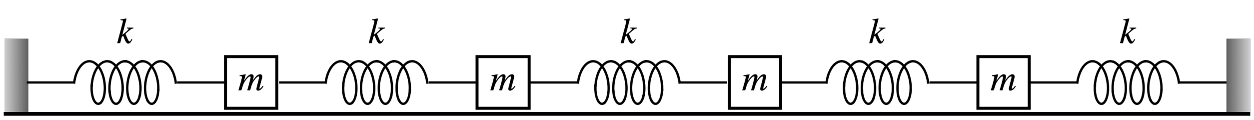
This theorem is critically important for quantum mechanics and the concept of simultaneous measurement.

(3) An "ellipsoid" is a three dimensional surface with three orthogonal symmetry axes, and which appears as an ellipse when viewed along any one axis. Show that the surface described by the points (x, y, z) that satisfy

$$5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4$$

is an ellipsoid. Find the directions of the axes of symmetry. Also, determine the lengths of the symmetry axes. (You are welcome to use MATHEMATICA to help find the eigenvalues and eigenvectors.)

(4) Find the four eigenfrequencies in terms of $\omega_0^2 \equiv k/m$, and describe the amplitudes for the normal modes to which they correspond, for the four masses connected by five springs on a frictionless horizontal surface, as shown below:



(5) For the system shown in Problem (4) above, find and plot the motions of each of the four masses as function of time, when all masses start from rest, with initial positions corresponding to each of the four normal modes. Most of the work for this problem is setting it up correctly in MATHEMATICA, identifying each eigenvector component with the correct mass and frequency.

PHY2502 Mathematical Physics Homework #11 Solutions

(1) The eigenvalue equation is $\underline{\underline{A}}\underline{x} = \lambda\underline{x}$. This can be written in terms of the transpose matrix and vector as $\underline{x}^{\mathsf{T}}\underline{\underline{A}}^{\mathsf{T}} = \lambda\underline{x}^{\mathsf{T}}$, and then taking the complex conjugate to get $\tilde{\underline{x}}\tilde{\underline{\underline{A}}} = \lambda^*\tilde{\underline{x}}$. Now multiply these two results together and find

$$\left(\tilde{\underline{x}}\tilde{\underline{\underline{A}}}\right)\left(\underline{\underline{A}}\underline{x}\right) = \tilde{\underline{x}}\left(\tilde{\underline{\underline{A}}}\underline{\underline{A}}\right)\underline{x} = \lambda^*\lambda^*\,\tilde{\underline{x}}\underline{x} \qquad \text{or} \qquad \langle x|x\rangle = |\lambda|^2\langle x|x\rangle$$

since $\tilde{\underline{\underline{A}}}\underline{\underline{A}} = \underline{\underline{I}}$. Therefore $|\lambda|^2 = 1$ which implies that $\lambda = e^{i\phi}$. For the given matrix

$$\tilde{\underline{\underline{A}}}\underline{\underline{A}} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the characteristic equation is $(i-\lambda)(-i-\lambda) = 1+\lambda^2 = 0$ so $\lambda = \pm i = e^{i\pi/2}, e^{3i\pi/2}$.

(2) For (a), suppose that $\underline{x}$ is an eigenvector of $\underline{\underline{A}}$ with eigenvalue λ, that is $\underline{\underline{A}}\underline{x} = \lambda\underline{x}$. Then

$$\underline{\underline{B}}\,(\lambda\underline{x}) = \underline{\underline{B}}\,\underline{\underline{A}}\underline{x} = \underline{\underline{A}}\,\underline{\underline{B}}\underline{x} = \underline{\underline{A}}\,\left(\underline{\underline{B}}\underline{x}\right) \qquad \text{and} \qquad \underline{\underline{B}}\,(\lambda\underline{x}) = \lambda\left(\underline{\underline{B}}\underline{x}\right) \qquad \text{so} \qquad \underline{\underline{A}}\,\left(\underline{\underline{B}}\underline{x}\right) = \lambda\left(\underline{\underline{B}}\underline{x}\right)$$

which is just a statement that $\underline{\underline{B}}\underline{x}$ is also an eigenvector of $\underline{\underline{A}}$. The eigenvectors of $\underline{\underline{A}}$ are unique, however, to within some scale factor. That is $\underline{\underline{B}}\underline{x} = \mu\underline{x}$, which means that $\underline{x}$ is also an eigenvector of $\underline{\underline{B}}$.

For (b), write $\underline{x}^{(i)}$ for a particular simultaneous eigenvector of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ where $\underline{\underline{A}}\,\underline{x}^{(i)} = \lambda^{(i)}\underline{x}^{(i)}$ and $\underline{\underline{B}}\,\underline{x}^{(i)} = \mu^{(i)}\underline{x}^{(i)}$. Now write, for an arbitrary vector $\underline{x}$,

$$\underline{x} = \sum_{i=1}^{N} c_i\underline{x}^{(i)}$$

and consider separately the actions of the matrices $\underline{\underline{A}}\,\underline{\underline{B}}$ and $\underline{\underline{B}}\,\underline{\underline{A}}$ on $\underline{x}$:

$$\begin{aligned}
\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{x} &= \underline{\underline{A}}\,\underline{\underline{B}}\sum_{i=1}^{N} c_i\underline{x}^{(i)} = \underline{\underline{A}}\sum_{i=1}^{N} c_i\underline{\underline{B}}\,\underline{x}^{(i)} = \underline{\underline{A}}\sum_{i=1}^{N} c_i\mu_i\underline{x}^{(i)} = \sum_{i=1}^{N} c_i\mu^{(i)}\underline{\underline{A}}\,\underline{x}^{(i)} = \sum_{i=1}^{N} c_i\mu^{(i)}\lambda^{(i)}\underline{x}^{(i)} \\
\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{x} &= \underline{\underline{B}}\,\underline{\underline{A}}\sum_{i=1}^{N} c_i\underline{x}^{(i)} = \underline{\underline{B}}\sum_{i=1}^{N} c_i\underline{\underline{A}}\,\underline{x}^{(i)} = \underline{\underline{B}}\sum_{i=1}^{N} c_i\lambda^{(i)}\underline{x}^{(i)} = \sum_{i=1}^{N} c_i\lambda^{(i)}\underline{\underline{B}}\,\underline{x}^{(i)} = \sum_{i=1}^{N} c_i\lambda^{(i)}\mu^{(i)}\underline{x}^{(i)}
\end{aligned}$$

These are the same result, since, of course, $\lambda^{(i)}\mu^{(i)} = \mu^{(i)}\lambda^{(i)}$. Therefore, $\underline{\underline{A}}\,\underline{\underline{B}}$ and $\underline{\underline{B}}\,\underline{\underline{A}}$ are the same since they give the same result when acting on an arbitrary vector.

(3) We need to diagonalize the symmetric matrix

$$\underline{\underline{A}} = \begin{bmatrix} 5 & -5 & 1 \\ -5 & 11 & -5 \\ 1 & -5 & 5 \end{bmatrix}$$

This is simple to do with MATHEMATICA. (I tried writing out by hand, but I couldn't see any easy way to factorize the characteristic polynomial.) The eigenvalues are $\lambda_1 = 16$, $\lambda_2 = 4$, and $\lambda_3 = 1$, so in the new coordinate system, the surface is

$$16x'^2 + 4y'^2 + z'^2 = 4$$

which is indeed an ellipsoid. The directions are given by the corresponding eigenvectors, namely

$$\underline{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Setting $x' = 0$ in the ellipsoid equation tells us that the ends are at $z' \pm 2$ and $y' = \pm 1$, while the ends in the other direction are $x' = \pm 1/2$, so the lengths of the axes are 4, 2, and 1.

(4) The setup for this problem is rather standard, giving the coupled equations of motion

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) = kx_1 - 2kx_2 + kx_3 \\ m\ddot{x}_3 &= -k(x_3 - x_2) + k(x_4 - x_3) = kx_2 - 2kx_3 + kx_4 \\ m\ddot{x}_4 &= -k(x_4 - x_3) - kx_4 = kx_3 - 2kx_4 \end{aligned}$$

Dividing through by m, inserting $x_i = a_i e^{i\omega t}$ and rearranging gives the eigenvalue problem

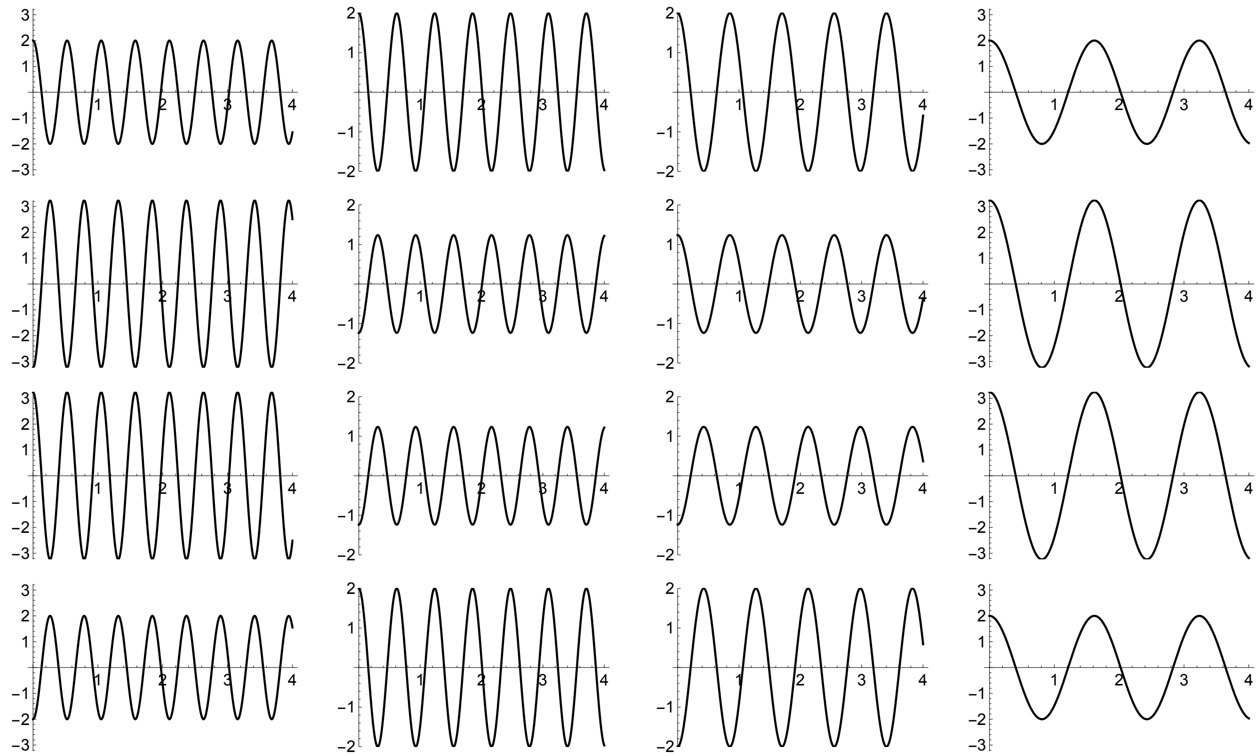
$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \qquad \text{where} \qquad \lambda = \frac{\omega^2}{\omega_0^2}$$

This is doable by hand, but see the MATHEMATICA notebook. You find $\lambda^{(1)} = (5 + \sqrt{5})/2$, $\lambda^{(2)} = (3 + \sqrt{5})/2$, $\lambda^{(3)} = (5 - \sqrt{5})/2$, and $\lambda^{(4)} = (3 - \sqrt{5})/2$, and the eigenvectors

$$\underline{a}^{(1)} = \begin{bmatrix} -1 \\ \frac{1}{2}\left(1+\sqrt{5}\right) \\ -\frac{1}{2}\left(1+\sqrt{5}\right) \\ 1 \end{bmatrix} \underline{a}^{(2)} = \begin{bmatrix} 1 \\ \frac{1}{2}\left(1-\sqrt{5}\right) \\ \frac{1}{2}\left(1-\sqrt{5}\right) \\ 1 \end{bmatrix} \underline{a}^{(3)} = \begin{bmatrix} -1 \\ \frac{1}{2}\left(1-\sqrt{5}\right) \\ -\frac{1}{2}\left(1-\sqrt{5}\right) \\ 1 \end{bmatrix} \underline{a}^{(4)} = \begin{bmatrix} 1 \\ \frac{1}{2}\left(1+\sqrt{5}\right) \\ \frac{1}{2}\left(1+\sqrt{5}\right) \\ 1 \end{bmatrix}$$

Note that $(1 + \sqrt{5})/2 \approx 1.6 > 1$ and $-1 < (1 - \sqrt{5})/2 \approx -0.6 < 0$, so the first mode has alternating signs with big amplitudes in the middle, the second mode is inners with small amplitude in synch with each other and out of synch with the outers, the third has the outer pairs in synch with each other with smaller amplitudes int he middle, and the fourth has all in synch with each other and larger amplitudes in the middle.

(5) We have the problem all set up and the eigenvalues and eigenvectors all determined, so use Mathematica to write the full solution out and apply the initial conditions. See the Mathematica notebook. Here are the plots, sorted left to right by eigenvalue, that is, the frequency of the normal mode:



These are clearly single-frequency oscillations, and match up well with what we determined for the eigenvalues and eigenvectors.

PHYS2502 Mathematical Physics Homework #12 Due 18 Apr 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

(1) Two horizontal identical circular hoops, each having radius R, are coaxial and separated vertically by a distance $2h$. A continuous soap film is attached to the hoops and drapes between them. Assuming that the surface tension of the film is proportional to its surface area, and that the film is in equilibrium when the surface tension is minimized, find the shape of the soap film. You can ignore the mass of the film, and you can leave your answer in terms of a single undetermined constant.

(2) A point particle of mass m moves in space as a function of time, following the position vector $\vec{r}(t) = \hat{\imath}\, x(t) + \hat{\jmath}\, y(t) + \hat{k}\, z(t)$. Assuming the particle moves according to the Principle of Least Action, that is, the path $\vec{r}(t)$ is the one that minimizes the functional

$$S[\vec{r}(t)] = \int_{t_1}^{t_2} L(x, y, z, \dot{x}, \dot{y}, \dot{z})\, dt \qquad \text{where} \qquad L = \frac{1}{2} m\dot{\vec{r}}^2 - V(\vec{r})$$

show that $\vec{F} = m\vec{a}$, aka "the equation of motion," where $\vec{F} = -\vec{\nabla}V$ and $\vec{a} = \ddot{\vec{r}}$.

(3) Two identical masses m slide on a frictionless horizontal surface and are each connected to a fixed outside wall and to each other by identical springs of stiffness k. The positions of the masses are given by $x_1(t)$ and $x_2(t)$. Knowing that the potential energy of a spring that is compressed or stretched a distance Δ is $k\Delta^2/2$, find the equation of motion for each of the two masses using the Principle of Least Action and the Euler-Lagrange equation. Check your answer against the example we have studied in class.

(4) A particle of mass m moves horizontally on a frictionless surface defined by the x, y plane. Convert to polar coordinates ρ and ϕ, and find the Euler-Lagrange equations of motion. Assuming that potential energy $V = V(\rho)$, that is it has no ϕ-dependence, show that one of the equations of motion leads to a "conserved quantity," that is, something that does not change with time. What is the common name for this conserved quantity?

(5) A highly flexible cable of linear mass density μ and fixed length ℓ hangs motionless in the vertical plane, where its shape minimizes the gravitational potential energy. The cable is fixed at two points at the same vertical position, but separated horizontally by a distance $d < \ell$. Assuming the shape of the cable is given by the function $f(x)$ where x measures the horizontal position, write the integrals that express (a) the gravitational potential energy and (b) the total length of the cable. Combine these integrals and use this to derive a constrained Euler-Lagrange equation that can be solved to find the shape of the hanging cable. Solve this equation for the shape. You don't need to get the result in terms of ℓ and d, but show how you would do that, in principle.

(1) Let the vertical position be measured by z, with $z = 0$ as the midpoint between the hoops, and $\rho = f(z)$ be the radial distance from the axis to the surface contour. An element ds of the surface area is

$$ds = 2\pi\rho\left[dz^2 + d\rho^2\right]^{1/2} = 2\pi\rho\left[1 + \left(\frac{d\rho}{dz}\right)^2\right]^{1/2} dz$$

Therefore we want to minimize

$$S[f(z)] = \int_{-h}^{h} 2\pi f(z)\left[1 + (f'(z))^2\right]^{1/2} dz = 2\pi\int_{-h}^{h} F(f(z), f'(z))\, dz$$

This is an example of the "second" special case, namely when F does not depend on z, so

$$\begin{aligned}
F - f'\frac{\partial F}{\partial f'} &= f(z)\left[1 + (f'(z))^2\right]^{1/2} - f'(z)\left\{\frac{f(z)f'(z)}{\left[1 + (f'(z))^2\right]^{1/2}}\right\} \\
&= \frac{f(z)}{\left[1 + (f'(z))^2\right]^{1/2}}\left\{1 + (f'(z))^2 - (f'(z))^2\right\} \\
&= \frac{f(z)}{\left[1 + (f'(z))^2\right]^{1/2}} = \text{constant} \equiv k
\end{aligned}$$

Manipulate this a little to get a manageable differential equation for $y = f(z)$.

$$y^2 = k^2(1 + y'^2) \qquad \text{so} \qquad \frac{dy}{dz} = \left(\frac{y^2}{k^2} - 1\right)^{1/2} \qquad \text{and} \qquad \frac{dy}{(y^2/k^2 - 1)^{1/2}} = dz$$

This can be integrated using $y = k\cosh u$ so $dy = k\sinh u\, du$ and $(y^2/k^2 - 1)^{1/2} = \sinh u$. The result is

$$ku = z + kc \qquad \text{so} \qquad k\cosh^{-1}\left(\frac{y}{k}\right) = x + kc \qquad \text{and} \qquad y = f(z) = k\cosh\left(\frac{z}{k} + c\right)$$

The symmetric boundary conditions $f(\pm h) = R$ imply that $c = 0$. The value of k would be set by some constraint on how much the soap film can stretch.

(2) It is helpful to write the Lagrangian as

$$L(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - V(x,y,z)$$

Now, there are three different Euler-Lagrange equations, one for each of the three dependent variables $x(t)$, $y(t)$, and $z(t)$. These are

$$\begin{aligned}
\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= 0 \qquad \text{or} \qquad -\frac{\partial V}{\partial x} - m\ddot{x} = 0 \\
\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} &= 0 \qquad \text{or} \qquad -\frac{\partial V}{\partial y} - m\ddot{y} = 0 \\
\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} &= 0 \qquad \text{or} \qquad -\frac{\partial V}{\partial z} - m\ddot{z} = 0
\end{aligned}$$

Using unit vectors to combine the second version of each of these equations, we get

$$-\hat{i}\frac{\partial V}{\partial x} - \hat{j}\frac{\partial V}{\partial y} - \hat{k}\frac{\partial V}{\partial z} = m\left[\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}\right] \qquad \text{that is} \qquad -\vec{\nabla}V = m\vec{a}$$

(3) In this case, the Lagrangian is

$$L(x_1,\dot{x}_1,x_2,\dot{x}_2) = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k(x_2-x_1)^2$$

The two Euler-Lagrange equations are therefore

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} &= \frac{\partial L}{\partial x_1} \qquad \text{so} \qquad m\ddot{x}_1 = -kx_1 - k(x_2-x_1)(-1) = -2kx_1 + kx_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} &= \frac{\partial L}{\partial x_2} \qquad \text{so} \qquad m\ddot{x}_1 = -kx_1 - k(x_2-x_1) = kx_1 - 2kx_2
\end{aligned}$$

which are the same two equations we ended up with before.

(4) The kinetic energy term is

$$\frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\left[\dot{x}^2+\dot{y}^2\right] = \frac{1}{2}m\left[(\dot{\rho}\cos\phi - \rho\sin\phi\,\dot{\phi})^2 + (\dot{\rho}\sin\phi + \rho\cos\phi\,\dot{\phi})^2\right] = \frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\phi}^2)$$

Therefore the Lagrangian is

$$L(\rho,\dot{\rho},\dot{\phi}) = \frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\phi}^2) - V(\rho)$$

which is independent of ϕ. The Euler-Lagrange equation for ϕ becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}m\rho^2\dot{\phi} = m\rho v_\perp = \text{constant}$$

which just says that angular momentum is conserved (for central potential energy functions).

(5) Position the y axis so that it is midway between the two fixed points at $x = \pm d/2$. Then

$$\begin{aligned} U[f(x)] &= \int_{-d/2}^{d/2} (\mu\, ds) g y = \mu g \int_{-d/2}^{d/2} f(x)\sqrt{1+f'(x)^2}\, dx \\ \text{and} \qquad \ell &= \int_{-d/2}^{d/2} ds = \int_{-d/2}^{d/2} \sqrt{1+f'(x)^2}\, dx \end{aligned}$$

Since $\delta\ell = 0$, we can find the minimum potential energy from

$$\delta U = \delta \left\{ \int_{-d/2}^{d/2} \left[\mu g f(x)\sqrt{1+f'(x)^2} + \lambda\sqrt{1+f'(x)^2} \right] dx \right\} = 0$$

which is your standard calculus of variations problem with the function

$$F(f, f') = \mu g f(x)\sqrt{1+f'(x)^2} + \lambda\sqrt{1+f'(x)^2}$$

This function is independent of x, so we have

$$\begin{aligned} F - f'\frac{\partial F}{\partial f'} &= \mu g f(x)\sqrt{1+f'(x)^2} + \lambda\sqrt{1+f'(x)^2} - f'(x)\frac{(\mu\, g f(x) + \lambda) f'(x)}{\sqrt{1+f'(x)^2}} \\ &= (\mu\, g f(x) + \lambda)\sqrt{1+f'(x)^2}\left[1 - \frac{f'(x)^2}{1+f'(x)^2}\right] \\ &= \frac{\mu\, g f(x) + \lambda}{\sqrt{1+f'(x)^2}} = \text{constant} \equiv c \end{aligned}$$

Writing $y = f(x)$, this gives, taking the positive square root,

$$(\mu g\, y + \lambda)^2 = c^2(1 + y'^2) \qquad \text{so} \qquad \frac{dy}{dx} = \sqrt{\frac{(\mu g\, y + \lambda)^2}{c^2} - 1}$$

This is easy to solve by putting $\mu g\, y + \lambda = c\cosh u$ in which case

$$\frac{dy}{dx} = \frac{c}{\mu g}\sinh u \frac{du}{dx} = \sqrt{\cosh^2 u - 1} = \sinh u \qquad \text{so} \qquad u = \frac{\mu g}{c}(x + k)$$

for some constant k. Therefore, the shape of the cable is given by

$$y = \frac{c}{\mu g}\left[\cosh\left(\frac{\mu g}{c}(x+k)\right) - \lambda\right]$$

In order to have $y = 0$ at $x = \pm d/2$ we need $k = 0$ and $\cosh(\mu g d/2c) = \lambda$ relates the values of c and λ. By integrating over the length of the cable, and setting it equal to ℓ, we would get another equation and determine separately their values.

PHYS2502 Mathematical Physics Homework #13 Due 25 Apr 2023

This homework assignment is due at the start of class on the date shown. Please submit a PDF of your solutions to the Canvas page for the course.

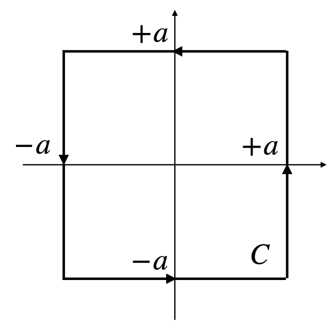
(1) If $z = x + iy$, where x and y are real numbers, then prove that the function $f(z) = e^z$ is analytic everywhere in the complex plane.

(2) If $z = x + iy$, where x and y are real numbers, then prove that the function $f(z) = 1/z$ is analytic everywhere in the complex plane except at $z = 0$.

(3) By direct integration, calculate the integral

$$\mathcal{I} = \int_C \frac{1}{z}\, dz$$

around the square contour with side length $2a$ shown here



and compare to the result you get from the Cauchy Integral Theorem.

(4) Evaluate the integral

$$\mathcal{I} = \int_{-\infty}^{\infty} \frac{e^{ikx}}{4x^2+1}\, dx$$

separately for the cases $k > 0$ and $k < 0$. Check your answers using MATHEMATICA.

(5) Consider two complex variables $w = u + iv$ and $z = x + iy$, and the "map" given by

$$w = z^2$$

For contours $u =$ constant in the w-plane, draw the contours to which they map in the z-plane. Repeat for contours $v =$ constant in the w-plane. Show that the two sets of contours in the z-plane are orthogonal to each other. That is, just as the contours for $u =$ constant and $v =$ constant are perpendicular to each other everywhere they intersect, so for the corresponding contours in the z-plane. For this reason, we refer to this mapping function as a "conformal map." Conformal maps have many applications in science and engineering.

(1) We have $f(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y = u(x,y) + iv(x,y)$. Now test the Cauchy-Riemann relations, namely

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y}$$

so the function is analytic everywhere.

(2) This one is a little more complicated. We write

$$f(z) = \frac{1}{z} = \frac{1}{z}\frac{z^*}{z^*} = \frac{x - iy}{x^2 + y^2} \qquad \text{so} \qquad u(x,y) = \frac{x}{x^2 + y^2} \qquad \text{and} \qquad v(x,y) = -\frac{y}{x^2 + y^2}$$

Now test the Cauchy-Riemann relations, namely

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} - \frac{x^2}{(x^2 + y^2)^2} = \frac{y^2}{(x^2 + y^2)^2} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = +\frac{y^2}{(x^2 + y^2)^2}$$

so the first relation is satisfied. We also need to check

$$\frac{\partial v}{\partial x} = +\frac{xy}{(x^2 + y^2)^2} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{xy}{(x^2 + y^2)^2}$$

and the second relation is satisfied. Of course, neither the function nor its derivatives are defined at $z = 0$, so the function is not analytic there.

(3) Symmetry argues that we should get the same result on each of the four sides of the square, so let's do the bottom leg where $y = \operatorname{Im} z = -a$ and we are integrating over $x = \operatorname{Re} z$ between $-a$ and $+a$:

$$\mathcal{I} = 4\int_{-a}^{a} \frac{1}{x + i(-a)}\, dx = 4\int_{-a}^{a} \frac{x + ia}{x^2 + a^2}\, dx = 4\int_{-a}^{a} \frac{x}{x^2 + a^2}\, dx + 4ia\int_{-a}^{a} \frac{1}{x^2 + a^2}\, dx$$

The first integral in the last expression is zero, since the integrand is an odd function. For the second term we make the substitution $x = a\tan\theta$ to get

$$\int_{-a}^{a} \frac{1}{x^2 + a^2}\, dx = \int_{-\tan^{-1}(x/a)}^{\tan^{-1}(x/a)} \frac{1}{a^2}\cos^2\theta \frac{dx}{d\theta} d\theta = \frac{1}{a^2}\int_{-\tan^{-1}(x/a)}^{\tan^{-1}(x/a)} a\, d\theta = \frac{2}{a}\tan^{-1}(1) = \frac{\pi a}{2}$$

Therefore $\mathcal{I} = 4ia(\pi a/2) = 2\pi i$. The Cauchy Integral Theorem says the result should be $2\pi i(1)$, so this is correct.

(4) This is very similar to the example worked in the textbook. We write

$$\mathcal{I} = \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+1/4} = \frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{ikx}}{(x+i/2)(x-i/2)}$$

so the poles are at $z_0 = \pm i/2$. For $k > 0$, we close the infinite semicircle in the upper plane so that the exponential goes to zero. This means that we pick up the pole at $z_0 = +i/2$ and the result is

$$\mathcal{I} = \frac{1}{4} 2\pi i \frac{e^{ik(i/2)}}{i/2+i/2} = \frac{\pi}{2}e^{-k/2}$$

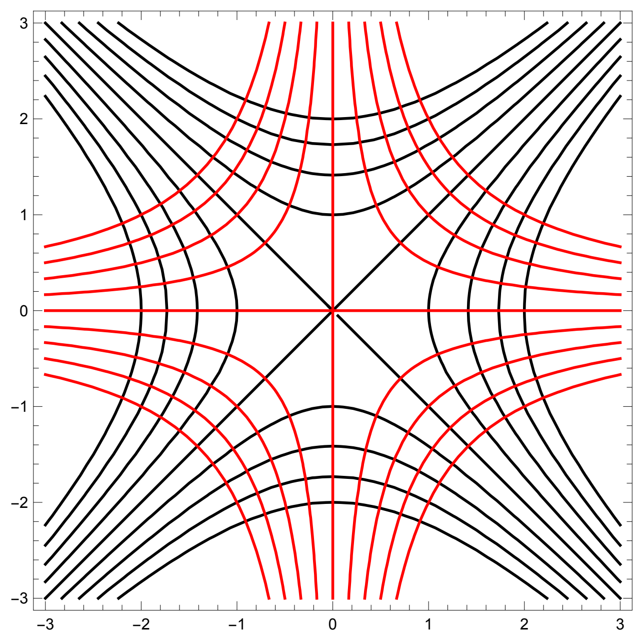
For $k < 0$ we close (in the clockwise direction) in the lower plane and pick up $z_0 = -i/2$, so

$$\mathcal{I} = -\frac{1}{4} 2\pi i \frac{e^{ik(-i/2)}}{-i/2-i/2} = \frac{\pi}{2}e^{k/2}$$

(5) This is pretty simple. First we have

$$w = (x+iy)^2 = x^2 - y^2 + 2ixy \qquad \text{so} \qquad u(x,y) = x^2 - y^2 \qquad \text{and} \qquad v(x,y) = 2xy$$

Contours of these functions are below, for u and v equal to the integers between -4 and 4, with u in black and v in red:



It certainly looks like the contours are perpendicular at all intersections, but let's prove it.

The slopes of two perpendicular lines have a product of -1. This is not hard to prove if you interpret the slope as the tangent of the angle with the horizontal, and compare the tangents of angles that differ by $\pi/2$, but let's prove it instead using the formalism we came up with in the book.

If a line is given by $x = m_x t + b_x$ and $y = m_y t + b_y$ then $(x - b_x)/m_x = (y - b_y)/m_y$ so writing the line in the form $y = mx + b$ says that the slope $m = m_y/m_x$. Now if two lines a

and b are perpendicular, then $m_x^{(a)} m_x^{(b)} + m_y^{(a)} m_y^{(b)} = 0$ so

$$\frac{m_y^{(b)}}{m_x^{(b)}} = -\frac{m_x^{(a)}}{m_y^{(a)}}$$

Therefore

$$m^{(a)} m^{(b)} = \frac{m_y^{(a)}}{m_x^{(a)}} \frac{m_y^{(b)}}{m_x^{(b)}} = -\frac{m_y^{(a)}}{m_x^{(a)}} \frac{m_x^{(a)}}{m_y^{(a)}} = -1$$

The tangents for the $u =$ constant curves are

$$m^{(u)} = \left.\frac{dy}{dx}\right|_{u=c} \qquad \text{and} \qquad 2x - 2y\frac{dy}{dx} = 0 \qquad \text{so} \qquad m^{(u)} = \frac{x}{y}$$

The tangents for the $v =$ constant curves are

$$m^{(v)} = \left.\frac{dy}{dx}\right|_{v=c} \qquad \text{and} \qquad 2y + 2x\frac{dy}{dx} = 0 \qquad \text{so} \qquad m^{(v)} = -\frac{y}{x}$$

and, indeed, $m^{(u)} m^{(v)} = -1$.