PHYS2502 Mathematical Physics: Spring 2023

Final Exam Tuesday 2 May 2023

There are **five questions** and you are to work all of them. You may use the textbook, your own notes, or any other resources, but you may not communicate with another human. Of course, if you have questions, you are encouraged to ask the person proctoring the exam.

The five problems will be equally weighted. If you are stuck on one, move on to another and come back if you have time.

Please start each problem on a new page in your exam booklet.

Good luck!

(1) The magnetic field $d\vec{B}$ from a short segment $d\vec{s}$ of an infinitely long straight wire C carrying a current I, at a point $\vec{r}$, measured relative to $d\vec{s}$, is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\vec{r}}{|\vec{r}|^3}$$

(a) What is the direction of the field $\vec{B}$ integrated over the length of the wire?

(b) Show that the magnitude of the integrated magnetic field at some point a distance R from the wire is given by $\mu_0 I/2\pi R$.

(c) Two long parallel wires, separated by a distance $2R$, each carry a current I but in opposite directions. If x measures the distance along an axis perpendicular to the wires, where $x = 0$ is the midpoint, find $B(x)$ for $|x| \ll R$, to the lowest nonzero order in x.

(2) Using the analytic techniques from class, solve for the function $y = f(x)$ where

$$\frac{dy}{dx} = e^{2x} + y - 1 \qquad \text{with} \qquad f(0) = 1$$

(3) Use contour integration to find the Fourier transform $A(k)$ of $f(x) = C/(x^2 + a^2)$, where C and a are positive constants. From inspection, make reasonable estimates for the widths of $f(x)$ and $A(k)$, and show that their product is independent of a.

(4) Find the directions of the symmetry axes of the three-dimensional ellipsoid given by

$$5x^2 + 5y^2 + 4z^2 + 2xy + 4xz + 4yz = 24$$

in the (x, y, z) coordinate system, and rewrite the ellipsoid equation in a coordinate system (x', y', z') that is rotated so that the primed axes align with the symmetry axes.

(5) Someone (with large hands) makes 100 throws of handfuls containing 11 eight-sided dice.

(a) How many throws should have exactly 3 dice landing with a "seven" facing up?

(b) What should be the average number of dice landing with a "seven" facing up?

(b) What should be the standard deviation on the average number of dice landing with a "seven" facing up?

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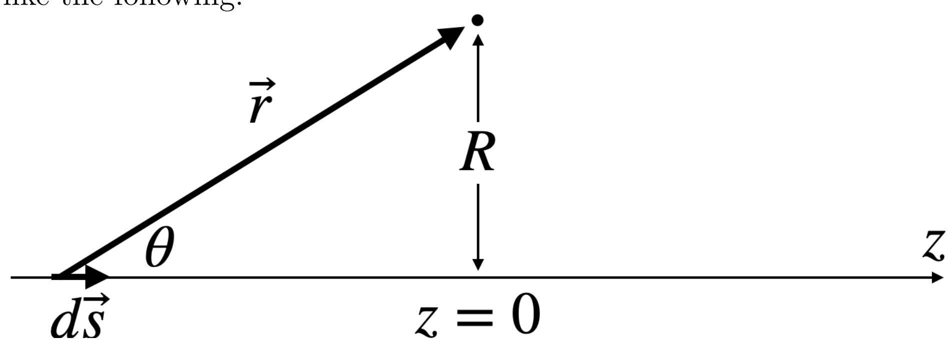
Solutions

(1) Part of the solution to this problem is to draw a good picture, which should look something like the following:



The vector $d\vec{s} \times \vec{r}$ is perpendicular to the wire, that is, perpendicular to $d\vec{s}$, and also perpendicular to $\vec{r}$, which means it points azimuthally around the wire. Its magnitude is $|d\vec{s}||\vec{r}| \sin\theta$ where θ is the angle between $d\vec{s}$ and $\vec{r}$. Therefore, if we let z measure the position along the wire with $z = 0$ being the point closest to the field point, then $|d\vec{s}| = dz$ and

$$B = \frac{\mu_0 I}{4\pi} \int_C \frac{|d\vec{s}||\vec{r}|}{|\vec{r}|^3} \sin\theta = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{1}{|\vec{r}|^2} \frac{R}{|\vec{r}|}\, dz = \frac{\mu_0 I R}{4\pi} \int_{-\infty}^{\infty} \frac{1}{(R^2 + z^2)^{3/2}}\, dz$$

The integral can be done with the substitution $z = R\tan\phi$ which gives

$$\begin{aligned} B &= \frac{\mu_0 I R}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{(R^2 + R^2 \tan^2\phi)^{3/2}} R \sec^2\phi\, d\phi = \frac{\mu_0 I}{4\pi R} \int_{-\pi/2}^{\pi/2} \cos\phi\, d\phi \\ &= \frac{\mu_0 I}{4\pi R} \sin\phi|_{-\pi/2}^{\pi/2} = \frac{\mu_0 I}{2\pi R} \end{aligned}$$

For the two long straight wires, the distance from one of them is $R + x$ and the distance from the other is $R - x$, and the fields are in opposite directions because the currents are in opposite directions, so

$$\begin{aligned} B &= \frac{\mu_0}{2\pi} \frac{I}{R - x} - \frac{\mu_0}{2\pi} \frac{I}{R + x} = \frac{\mu_0}{2\pi R} \left[\frac{1}{1 - x/R} - \frac{1}{1 + x/R} \right] \\ &\approx \frac{\mu_0}{2\pi R} \left[1 + \frac{x}{R} - \left(1 - \frac{x}{R} \right) \right] = \frac{\mu_0}{\pi R^2} x \end{aligned}$$

(2) This equation is in the form we studied for solutions using an integrating factor, that is

$$\frac{dy}{dx} + p(x)y = g(x) \qquad \text{where} \qquad p(x) = -1 \qquad \text{and} \qquad g(x) = e^{2x} - 1$$

The integrating factor is

$$\mu(x) = \exp\left[\int p(x)\,dx\right] = e^{-x}$$

so we have

$$\frac{d}{dx}\left(ye^{-x}\right) = e^{-x}g(x) = e^{x} - e^{-x}$$

Integrating both sides and applying the boundary condition gives

$$ye^{-x} = e^{x} + e^{-x} + c \qquad \text{with} \qquad 1 = 2 + c \qquad \text{so} \qquad c = -1$$

and the final solution is

$$y = f(x) = e^{2x} + 1 - e^{x}$$

(3) Apply the definition of the Fourier transform to get

$$A(k) = \int_{-\infty}^{\infty} C\frac{e^{-ikx}}{x^2 + a^2}\,dx = \int_{-\infty}^{\infty} C\frac{e^{-ikx}}{(x - ia)(x + ia)}\,dx$$

The integral is straightforward using the Cauchy integral theorem. if $k > 0$ then close the (clockwise) contour in the lower plane so that the exponential goes to zero along the semicircle, in which case you pick up the pole at $z = -ia$. This gives

$$A(k) = (-)2\pi iC\,\frac{e^{-ka}}{-ia - ia} = \frac{\pi C}{a}e^{-ka} \qquad k > 0$$

For $k < 0$, we close in the upper plane and pick up $z = ia$ and get

$$A(k) = 2\pi iC\,\frac{e^{ka}}{ia + ia} = \frac{\pi C}{a}e^{ka} \qquad k < 0$$

In other words

$$A(k) = \frac{\pi C}{a}e^{-|k|a}$$

Since $f(x)$ falls by 1/2 at $x = \pm a$, call the width of $f(x)$ as $\Delta f = 2a$. Since $A(k)$ falls by $1/e$ when $ka = \pm 1$, call the width of $A(k)$ as $\Delta A = 2/a$. So $\Delta f\,\Delta A = 4$, independent of a.

(4) This problem is worked almost identically to HW #11 Problem 3, and Lab #11. We need to diagonalize the (symmetric) matrix of coefficients given by

$$\underline{\underline{A}} = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

The problem can be solved from here with MATHEMATICA, but the numbers work out so that it is not hard to calculate by hand. The characteristic equation for the eigenvalues λ is

$$\begin{aligned} & (5-\lambda)^2(4-\lambda) + 4 + 4 - 4(5-\lambda) - 4(5-\lambda) - (4-\lambda) \\ = \; & (5-\lambda)^2(4-\lambda) + 8 - 8(5-\lambda) - (4-\lambda) \\ = \; & (5-\lambda)^2(4-\lambda) - 36 + 9\lambda \\ = \; & (5-\lambda)^2(4-\lambda) - 9(4-\lambda) = (4-\lambda)(25 - 10\lambda + \lambda^2 - 9) \\ = \; & (4-\lambda)(16 - 10\lambda + \lambda^2) = (4-\lambda)(2-\lambda)(8-\lambda) = 0 \end{aligned}$$

So the eigenvalues are $\lambda^{(1)} = 2$, $\lambda^{(2)} = 4$, and $\lambda^{(3)} = 8$. For $\lambda = \lambda^{(1)}$ we have

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{i.e.} \qquad \begin{aligned} 3x + y + 2z &= 0 \\ x + 3y + 2z &= 0 \\ x + y + z &= 0 \end{aligned}$$

Subtracting the second equation from the first gives $x = y$, which means the third eauation gives $z = -2x = -2y$, so $\vec{v}^{(1)} = 1\hat{\imath} + 1\hat{\jmath} - 2\hat{k}$.

For $\lambda = \lambda^{(2)}$ we have

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{i.e.} \qquad \begin{aligned} x + y + 2z &= 0 \\ x + y + 2z &= 0 \\ x + y &= 0 \end{aligned}$$

so $y = -x$ and $z = 0$ and $\vec{v}^{(2)} = 1\hat{\imath} - 1\hat{\jmath}$.

For $\lambda = \lambda^{(3)}$ we have

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{i.e.} \qquad \begin{aligned} -3x + y + 2z &= 0 \\ x - 3y + 2z &= 0 \\ x + y - 2z &= 0 \end{aligned}$$

Subtract the first two equations to get $-4x + 4y = 0$ or $y = x$, in which case the third equation gives $z = x = y$ and therefore $\vec{v}^{(3)} = 1\hat{\imath} + 1\hat{\jmath} + 1\hat{k}$.

Of course, the eigenvalues just give you the coefficients of the ellipsoid in the primed coordinates, so you get

$$2x'^2 + 4y'^2 + 8z'^2 = 24$$

for the ordering of the eigenvectors that I assumed above.

(5) This is an application of the binomial distribution, where, using the notation we used in class, $n = 11$, $m = 3$, and $p = 1/8$. The expected number of throws is therefore

$$100 \times \left(\begin{array}{c} 11 \\ 3 \end{array} \right) \left(\frac{1}{8}\right)^3 \left(\frac{7}{8}\right)^8 = 100 \times \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} \frac{7^8}{8^{11}} = 100 \times (11 \times 15) \frac{7^8}{8^{11}} = 11.0733$$

The average number of dice that should land with a "seven" (or any particular number, for that matter) is $\mu = np = 11/8 = 1.375$.

The standard deviation σ on the average number of dice that should land with a "seven" (or any particular number, for that matter) is given by $\sigma^2 = np(1-p) = 11(1/8)(7/8) = 77/64$. Therefore $\sigma = 1.10$.