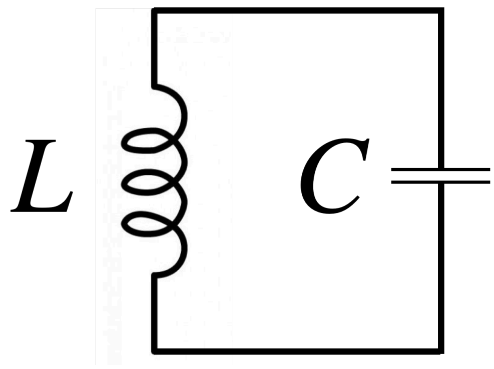
PHYS2063 Wave Physics Homework #1 Due Thursday 25 Aug 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

This diagram shows an inductor L and a capacitor C connected in series:



The voltage drop across an inductor is $V_L = LdI/dt$ where I is the current flowing through the inductor. The voltage drop across a capacitor is $V_C = q/C$ where q is the charge on the capacitor plates. Kirchoff's voltage law says that the sum of all voltage drops around a closed loop is zero. You can assume that the wires connecting the inductor and capacitor have no resistance. The charge (and current) are functions of time t.

Find the differential equation for the charge $q(t)$ and show that it implies that $q(t)$ has a simple harmonic time dependence. Find the (angular) frequency ω in terms of L and C for these oscillations.

Assume the values $L = 0.1$ mH and $C = 1$ μF. At time $t = 0$ the current in the circuit is $I_0 = 100$ μA and the charge on the capacitor is $q_0 = 1$ nC. Find the amplitude and phase of the oscillations for $q(t)$.

Homework #1 Solutions

The current $I(t) = dq/dt = \dot{q}$, so the voltage drop across L is $L\ddot{q}$ and Kirchoff's law gives

$$L\ddot{q} + \frac{q}{C} = 0 \qquad \text{or} \qquad \ddot{q} + \omega^2 a = 0 \qquad \text{where} \qquad \omega^2 = \frac{1}{LC}$$

The angular frequency, sticking with SI units, is

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-4} \times 10^{-6}}} = 10^5 \text{ Hz}$$

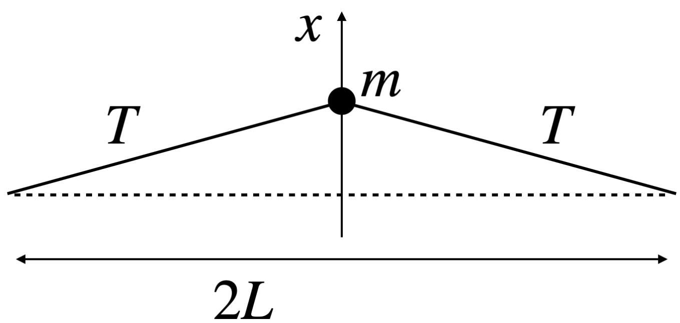
Write $q(t) = A\cos(\omega t + \phi)$, so $I(t) = \dot{q}(t) = -\omega A \sin(\omega t + \phi)$. Then, again with SI units,

$$\begin{aligned} q_0 &= A\cos\phi \\ I_0 &= -\omega A \sin\phi \\ \tan\phi &= -\frac{I_0}{\omega q_0} = -\frac{10^{-4}}{10^5 \times 10^{-9}} = -1 \qquad \text{so} \qquad \phi = -\frac{\pi}{4} = -45^\circ \\ A &= \sqrt{q_0^2 + \left(\frac{I_0}{\omega}\right)^2} = \sqrt{10^{-18} + \left(\frac{10^{-4}}{10^5}\right)^2} = \sqrt{2} \text{ nC} \end{aligned}$$

PHYS2063 Wave Physics Homework #2 Due Tuesday 30 Aug 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

(1) A bead of mass m sits in the middle of a string with unstretched length $2L$.



The string has tension T and the vertical position of the mass is measured by x. Find the frequency of oscillations for the mass along the x axis, assuming that the amplitude is much smaller than $2L$.

(2) Use Euler's formula to prove that

$$\cos(x+y) = \cos(x)\cos(y) - \sin(y)\sin(x)$$
and
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

(3) An object moves in the xy plane according to the equations

$$x(t) = A_x \cos(\omega t + \phi_x) \qquad \text{and} \qquad y(t) = A_y \cos(\omega t + \phi_y)$$

Plot and describe the shapes of the trajectories when

- $A_x = A_y$ and $\phi_x = \phi_y$
- $2A_x = A_y$ and $\phi_x = \phi_y$
- $2A_x = A_y$ and $\phi_x - \phi_y = \pi/4$

If you make the plots in Mathematica, the function you want to use is ParametricPlot.

(4) Show that, for the series LC oscillator,

$$\frac{d}{dt}\left[\frac{q^2}{2C} + \frac{1}{2}L\dot{q}^2\right] = 0$$

and interpret the meanings of the two terms in square brackets.

Homework #2 Solutions

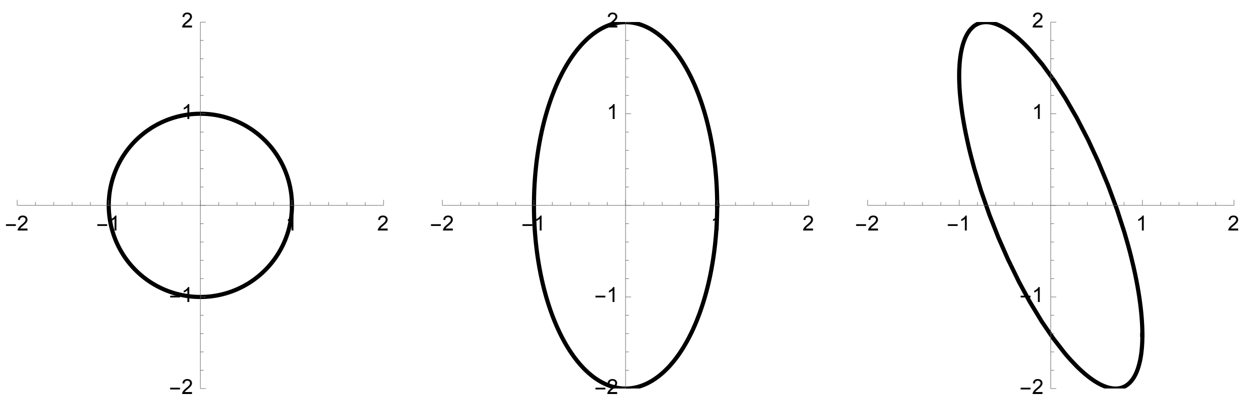
(1) There is only a force acting in the x-direction because the horizontal components cancel. If θ is the angle the string makes with the horizontal, then the restoring force is

$$F = 2 \times T \sin\theta = 2T\frac{x}{L} \qquad \text{so} \qquad m\ddot{x} = \frac{2T}{L}x \qquad \text{giving} \qquad \omega^2 = \frac{2T}{mL}$$

(2) Just write $e^{i(x+y)} = e^{ix}e^{iy}$ and write out both sides using Euler's formula, and equate the real and imaginary parts.

$$\begin{aligned} \cos(x+y) + i\sin(x+y) &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= \cos(x)\cos(y) - \sin(y)\sin(x) + i[\sin(x)\cos(y) + \cos(x)\sin(y)] \end{aligned}$$

(3) See the MATHEMATICA notebook. Here are the plots:



This can also be done algebraically, the the equation of the tilted ellipse is not so obvious.

(4) Just take the derivative and factor out $\dot{q}$:

$$\frac{d}{dt}\left[\frac{q^2}{2C} + \frac{1}{2}L\dot{q}^2\right] = \frac{q\dot{q}}{C} + L\dot{q}\ddot{q} = \dot{q}\left(\frac{q}{C} + L\ddot{q}\right) = 0$$

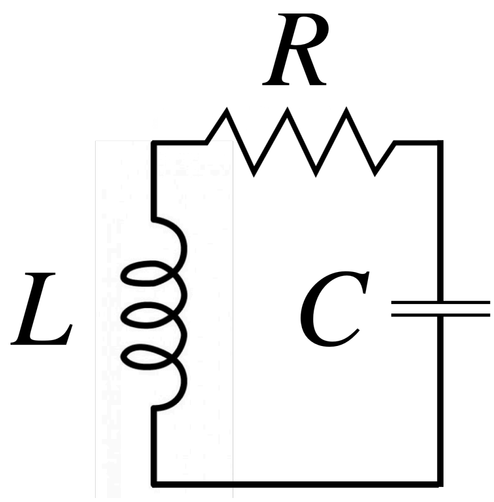
where the last step just makes use of the circuit equation we got in the first homework assignment.

This conserved quantity is just the energy in the circuit. The first term is the energy stored in the capacitor, and the second term is the energy stored in the inductor.

PHYS2063 Wave Physics Homework #3 Due Thursday 1 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

We now add a resistor R to our series circuit of an inductor and capacitor:



The voltage drop across a resistor with a current I flowing through it is IR. Note that unlike inductors and capacitors, resistors do not store energy.

Use this information, and what you've already done, to find the rate at which energy is dissipated by the resistor, in terms of I and R. You will almost certainly recognize the answer from your introductory physics course.

The differential equation is now

$$L\frac{dI}{dt} + \frac{q}{C} + IR = 0$$

Therefore the rate of change of energy in the circuit is

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{q^2}{2C} + \frac{1}{2}L\dot{q}^2\right] = \frac{q\dot{q}}{C} + L\dot{q}\ddot{q} = I\left(\frac{q}{C} + L\dot{I}\right) = -I^2R$$

PHYS2063 Wave Physics   Homework #4   Due Tuesday 6 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

(1) We derived a differential equation in class for the damped oscillator, namely

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$$

Consider the "critical damping" case where $\beta = \omega_0$. Substitute $x(t) = u(t)e^{-\beta t}$ to find, and then solve, a differential equation for $u(t)$. Apply the initial conditions to find the complete solution for $x(0) = x_0$ and $\dot{x}(0) = v_0$.

(2) Make plots of the motion $x(t)$ for the damped oscillator for the initial conditions $x_0 = 2$ and $v_0 = 0$, for each of the three cases

(a) $\beta = \omega_0/5$
(b) $\beta = 5\omega_0$
(c) $\beta = \omega_0$

Plot as a function of the time t in units of $2\pi/\omega_0$.

(3) Consider a (linear) lightly damped mechanical oscillator with mass m, spring constant k, and damping coefficient b, where $b/m \ll \sqrt{k/m}$. Find an expression for the fractional decrease in energy over one period of oscillation, in terms of m, k, and b. (It is probably easiest to do this by considering how the energy depends on the amplitude, and then finding the fractional decrease in the amplitude.)

Also express your answer in terms of $\beta = b/2m$ and $\omega_0^2 = k/m$.

The inverse of this fraction is often written as $Q/2\pi$ where Q is called the *Quality Factor*.

(4) Find an expression for Q in terms of L, C, and R for the series LCR circuit in the previous homework assignment.

Homework #4 <u>Solutions</u>

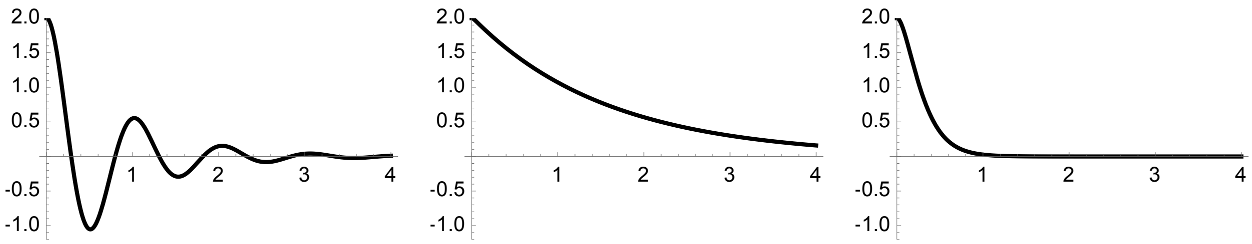
(1) We have $\dot{x}(t) = [\dot{u}(t) - \beta u(t)]e^{-\beta t}$ and $\ddot{x}(t) = [\ddot{u}(t) - 2\beta\dot{u}(t) + \beta^2 u(t)]e^{-\beta t}$, so

$$\begin{aligned}\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) &= [\ddot{u}(t) - 2\beta\dot{u}(t) + \beta^2 u(t) + 2\beta\dot{u}(t) - 2\beta^2 u(t) + \omega_0^2 u(t)]e^{-\beta t} \\ &= [\ddot{u}(t) + \beta^2 u(t) - 2\beta^2 u(t) + \omega_0^2 u(t)]e^{-\beta t} = \ddot{u}(t)e^{-\beta t} = 0\end{aligned}$$

Therefore $\ddot{u}(t) = 0$ so $u(t) = a + bt$ and

$$\begin{aligned}x(t) &= (a+bt)e^{-\beta t} \\ x(0) &= a = x_0 \\ \dot{x}(t) &= be^{-\beta t} - \beta(a+bt)e^{-\beta t} \\ \dot{x}(0) &= b - \beta a = b - \beta x_0 = v_0 \qquad \text{so} \qquad b = \beta x_0 + v_0\end{aligned}$$

(2) All calculations done in the Mathematica notebook.



(3) The energy of the oscillator is $E(t) = kA^2(t)/2$ where $A(t) = A_0 e^{-\beta t} = A_0 e^{-bt/2m}$. Since $b/m \ll \omega_0$, one period of oscillation is very close to $T = 2\pi/\omega_0$. Therefore, the fractional change in the energy over one oscillation period is

$$\frac{E(0) - E(T)}{E(0)} = \frac{A^2(0) - A^2(T)}{A^2(0)} = 1 - e^{-2\pi b/m\omega_0} \approx 1 - 1 + \frac{2\pi b}{m\omega_0} = \frac{2\pi b}{m\omega_0} = \frac{2\pi b}{\sqrt{km}}$$

In other words $Q/2\pi = m\omega_0/2\pi b$ so $Q = \omega_0/2\beta$.

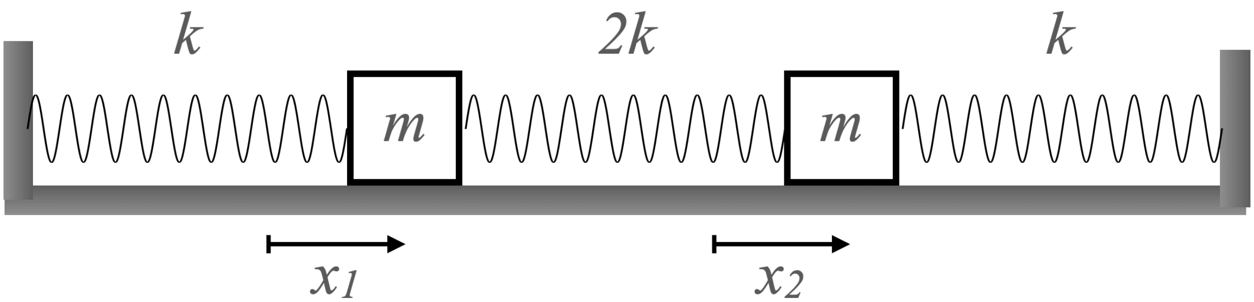
(4) We still have $Q = \omega_0/2\beta$ but now $\omega_0 = 1/\sqrt{LC}$ and $\beta = R/2L$ so

$$Q = \frac{1}{\sqrt{LC}}\frac{L}{R} = \sqrt{\frac{L}{R^2 C}}$$

PHYS2063 Wave Physics Homework #5 Due Thursday 8 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

Two masses m are connected by three springs, where the middle spring has twice the stiffness of the outer springs:



Write down the differential equations of motion for $x_1(t)$ and $x_2(t)$. Find the two fundamental frequencies in terms of $\omega_0^2 = k/m$.

Just consider the forces on the two masses and go from there.

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + 2k(x_2 - x_1) = -3kx_1 + 2kx_2 \\ m\ddot{x}_2 &= -kx_2 - 2k(x_2 - x_1) = 2kx_1 - 3kx_2 \end{aligned}$$

Rewrite in terms of ω_0^2 and insert $x = ae^{i\omega t}$ to get

$$\begin{aligned} -\omega^2 a_1 &= -3\omega_0^2 a_1 + 2\omega_0^2 a_2 \\ -\omega^2 a_2 &= 2\omega_0^2 a_1 - 3\omega_0^2 a_2 \end{aligned}$$

Now rewrite as algebraic equations for a_1 and a_2.

$$\begin{aligned} (3\omega_0^2 - \omega^2)a_1 - 2\omega_0^2 a_2 &= 0 \\ -2\omega_0^2 a_1 + (3\omega_0^2 - \omega^2)a_2 &= 0 \end{aligned}$$

Finally, set the determinant equal to zero and solve for ω^2.

$$\begin{aligned} (3\omega_0^2 - \omega^2)^2 &= (2\omega_0^2)^2 \\ \omega^2 - 3\omega_0^2 &= \pm 2\omega_0^2 \end{aligned}$$

and the two frequencies are $\omega^2 = \omega_0^2$ (when the middle spring keeps its length constant) and $\omega^2 = 5\omega_0^2$ (when the two masses oscillate opposite to each other).

PHYS2063 Wave Physics Homework #6 Due Tuesday 13 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

Continue with the last homework problem, and find the motion for general initial conditions $x_1(0) = x_{1_0}$, $\dot{x}_1(0) = v_{1_0}$, $x_2(0) = x_{2_0}$, and $\dot{x}_2(0) = v_{2_0}$. See the *Mathematical Concepts* book, Section 3.7. Make plots similar to Figure 3.13 showing the motion of the fundamental modes.

Homework #6 <u>Solutions</u>

See the MATHEMATICA notebook.

PHYS2063 Wave Physics Homework #7 Due Thursday 15 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

In class we derived the following formula for the fundamental frequencies for transverse planar vibrations of the loaded string with n degrees of freedom:

$$\omega_j^2 = 2\omega_0^2 \left[1 - \cos\left(\frac{j\pi}{n+1}\right)\right] \qquad j = 1, 2, 3, \ldots, n$$

Show that this formula reduces to the results we derived in class last week for $n = 2$ and $n = 3$, when we analyzed a system of n masses connected by $n+1$ springs, oscillating in one dimension.

For $n = 2$, we consider $j = 1$ and $j = 2$, so

$$\begin{aligned}
\omega_1^2 &= 2\omega_0^2\left[1-\cos\frac{\pi}{3}\right] = 2\omega_0^2\left[1-\frac{1}{2}\right] = \omega_0^2 \\
\omega_2^2 &= 2\omega_0^2\left[1-\cos\frac{2\pi}{3}\right] = 2\omega_0^2\left[1-\left(-\frac{1}{2}\right)\right] = 3\omega_0^2
\end{aligned}$$

which in fact are the results we derived in class, and demonstrated with the carts and springs.

For $n = 3$, we have

$$\begin{aligned}
\omega_1^2 &= 2\omega_0^2\left[1-\cos\frac{\pi}{4}\right] = 2\omega_0^2\left[1-\frac{\sqrt{2}}{2}\right] = (2-\sqrt{2})\omega_0^2 \\
\omega_2^2 &= 2\omega_0^2\left[1-\cos\frac{2\pi}{4}\right] = 2\omega_0^2\left[1-0\right] = 2\omega_0^2 \\
\omega_3^2 &= 2\omega_0^2\left[1-\cos\frac{3\pi}{4}\right] = 2\omega_0^2\left[1-\left(-\frac{\sqrt{2}}{2}\right)\right] = (2+\sqrt{2})\omega_0^2
\end{aligned}$$

which also agree with what we derived in class.

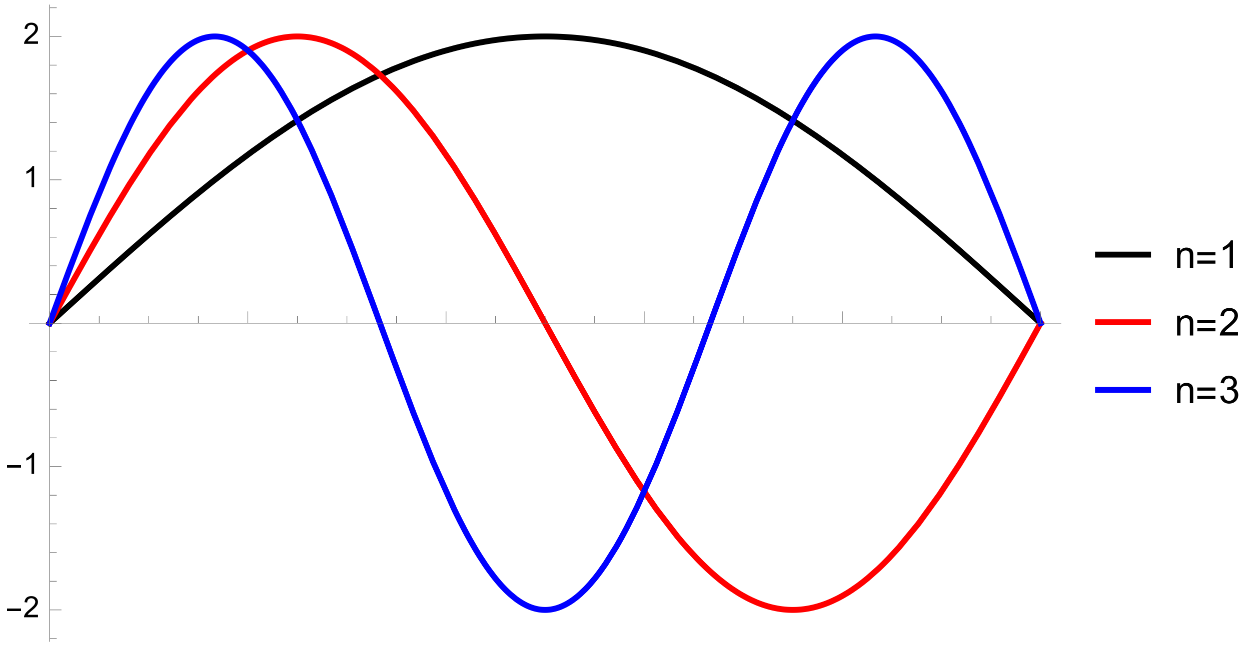
PHYS2063 Wave Physics Homework #8 Due Tuesday 20 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

Make plots of the first three fundamental modes of a stretched string of length L. Give all three modes the same amplitude.

This is a simple exercise. However, I strongly encourage you to solve it with MATHEMATICA. We will be building on this when we do more work to understand standing waves on a stretched string. Those exercises will be much easier using MATHEMATICA, and this exercise is the starting point.

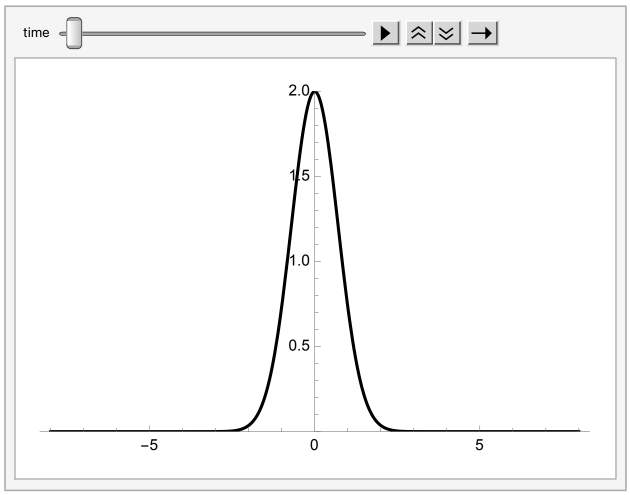
See the MATHEMATICA notebook. Here is the plot:



PHYS2063 Wave Physics Homework #9 Due Thursday 22 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

A Gaussian pulse shape $f(x) = Ae^{-x^2}$ starts from rest on a stretched string. Make an animation that shows how the pulse splits in two, with half going to the right and half going to the left. Your animation should look something like the following at the start:



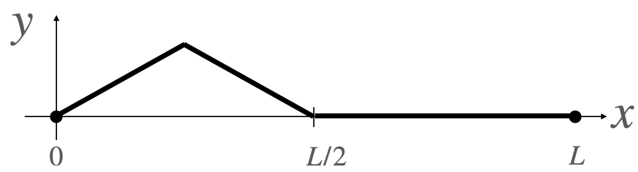
Homework #9 <u>Solutions</u>

See the MATHEMATICA notebook.

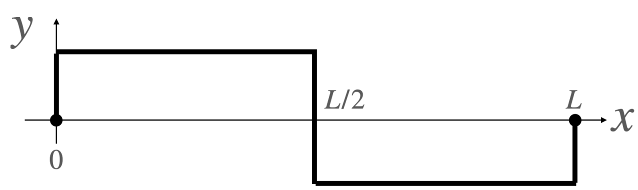
PHYS2063 Wave Physics Homework #10 Due Tuesday 27 Sep 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

Following the examples we did in class, and also the posted MATHEMATICA notebook, generate the Fourier Sine Series approximations for the two functions depicted here. The first is a simple triangle on the left half of the string, and flat on the right half:



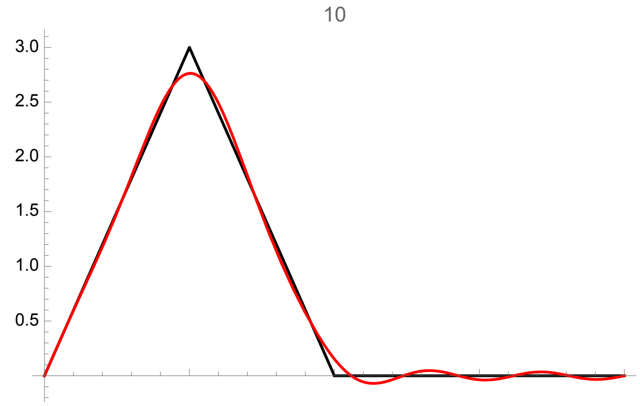
The second example is a "square wave" of one period. Assume that the string is indeed fixed to the x-axis at both $x = 0$ and $x = L$:

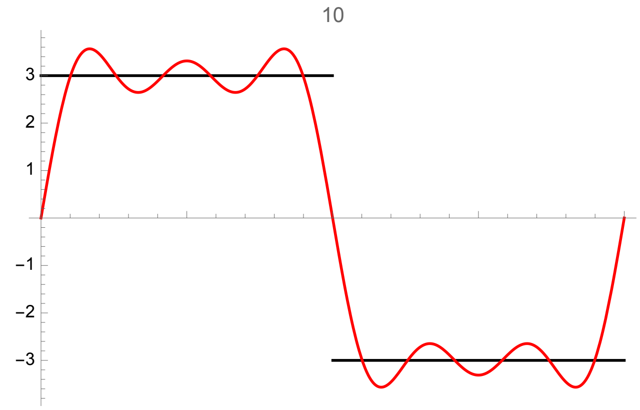


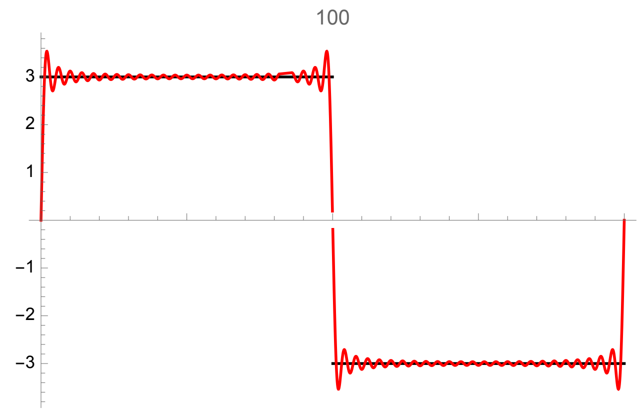
Make plots, as we did in class, with different values for the maximum number Fourier sine terms, to get a sense of how many terms you need to get a good convergence to the right answer, in each case.

Homework #10 <u>Solutions</u>

See the Mathematica notebook. Here are some plots. The number at the top tells the value of nMax.







PHYS2063 Wave Physics Homework #11 Due Tuesday 4 Oct 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

These are simple exercises to give you some practice in vector operations. I am pretty sure that you've already seen this material in your calculus classes. You might find it handy to refer to Chapter Four in the "Concepts" book.

I think it is best if you do these problems by hand, although you can check your answers with MATHEMATICA if you like.

(1) Find a vector $\vec{v}$ that is perpendicular to the vector $\vec{w} = 2\hat{\imath} + 1\hat{\jmath}$, also lying in the xy plane, and that has unit length.

(2) Find a vector $\vec{v}$ that is perpendicular to the vectors $\vec{u} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{w} = 2\hat{\imath} + 4\hat{\jmath} - 3\hat{k}$, and that has unit length.

For the following problems, use the gradient operator in Cartesian coordinates, that is

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

(3) Find the gradient $\vec{\nabla}F$ for the field $F(x, y, z) = x^2yz^{1/2}$.

(4) Find the divergence $\vec{\nabla} \cdot \vec{E}$ for the vector field

$$\vec{E}(x, y, z) = \frac{\hat{\imath}\,x + \hat{\jmath}\,y + \hat{k}\,z}{(x^2 + y^2 + z^2)^{3/2}}$$

(5) Find the curl $\vec{\nabla} \times \vec{B}$ for the vector field

$$\vec{B}(x, y, z) = -\hat{\imath}\,\frac{y}{x^2 + y^2} + \hat{k}\,\frac{x}{x^2 + y^2}$$

(1) Write $\vec{v} = a\hat{i} + b\hat{j}$. Then $\vec{v} \cdot \vec{w} = 0$ implies that $2a + b = 0$ or $b = -2a$. Unit length means that $a^2 + b^2 = 5a^2 = 1$. Therefore $a = \pm 1/\sqrt{5}$. Picking the positive solution, we have

$$\vec{v} = \sqrt{\frac{1}{5}}\,\hat{i} - 2\sqrt{\frac{1}{5}}\,\hat{j}$$

(2) Just take the cross product to get a vector that is perpendicular to both. That is

$$\vec{u} \times \vec{w} = (3-8)\hat{i} + (4+3)\hat{j} + (4+2)\hat{k} = -5\hat{i} + 7\hat{j} + 6\hat{k}$$

The sum of squares of the components is $25 + 49 + 36 = 110$, so

$$v = -\frac{5}{\sqrt{110}}\,\hat{i} + \frac{7}{\sqrt{110}}\,\hat{j} + \frac{6}{\sqrt{110}}\,\hat{k}$$

(3) $\vec{\nabla} F = \hat{i}\,2xyz^{1/2} + \hat{j}\,x^2 z^{1/2} + \hat{k}\,x^2 y/2z^{1/2}$.

(4) We have $\vec{\nabla} \cdot \vec{E} = \partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z$. We calculate

$$\frac{\partial E_x}{\partial x} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{x}{(x^2+y^2+z^2)^{5/2}}2x = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

so it is clear that $\vec{\nabla} \cdot \vec{E} =]3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2]/(x^2+y^2+z^2)^{5/2} = 0$, except, of course, at the origin.

(5) There is no component B_z, and neither B_x nor B_y depend on z, so from (4.20a) in Concepts,

$$\begin{aligned}
\vec{\nabla} \times \vec{B} &= \hat{k}\left[\frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right] \\
&= \hat{k}\left[\frac{1}{2}\frac{2xy}{(x^2+y^2)^2} - \frac{1}{2}\frac{2yx}{(x^2+y^2)^2}\right] = 0
\end{aligned}$$

except, of course, at the origin.

PHYS2063 Wave Physics Homework #12 Due Thursday 6 Oct 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

Use MATHEMATICA to combine a "contour" plot of the scalar field

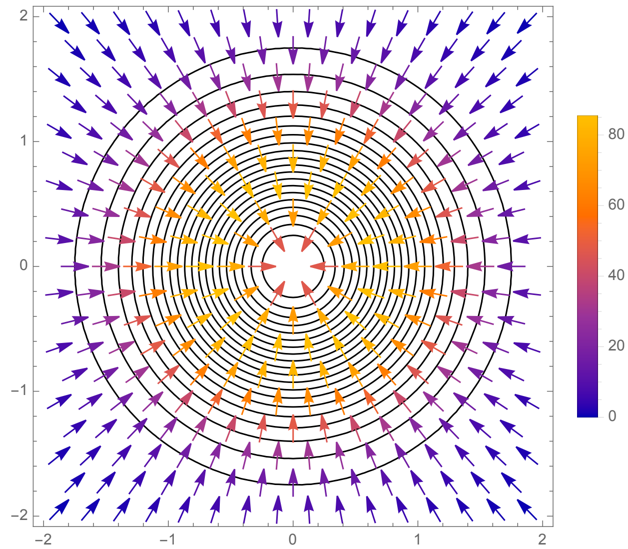
$$f(x, y, z) = 100\, e^{-(x^2+y^2)}$$

and a "vector" plot of $\vec{\nabla} f$. Make the x and y axis ranges large enough to show several contours. This will let you observe how the gradient is larger where the contours are closer together.

The functions you'll probably want to use are ContourPlot and VectorPlot. I think that the option ContourShading → None is best. You might try using Contours → N to get N contours. Use the option PlotLegends → Automatic to include the meaning of the colors in the vector plot.

Homework #12 | Solutions

See the MATHEMATICA notebook. Here is my plot.



PHYS2063 Wave Physics Homework #13 Due Tuesday 11 Oct 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

(1) In addition to Gauss' Theorem and Stokes' Theorem, there is a third "surface theorem" which, for some reason, doesn't find its way much into physics problems. The theorem is

$$\int_V \vec{\nabla} f \, dV = \oint_S f \, d\vec{S}$$

where $f(x, y, x)$ is some scalar field, and S is the surface enclosing V. Prove this using Gauss' Theorem and the vector field $\vec{A} = \vec{C} f$ where $\vec{C}$ is some arbitrary constant vector.

(2) Show that Gauss' Theorem holds for the vector field $\vec{A} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$ and the cubic volume V with side length L in the first octant (x, y, and z all positive) with one corner at the origin.

(3) Show that Stokes' Theorem holds for the vector field $\vec{A} = -\hat{\imath}y + \hat{\jmath}x$ and the square surface S with side length L in the first quadrant (x and y both positive) with one corner at the origin.

(4) Two homework assignments ago, you calculated the divergence of the vector field

$$\vec{E}(x, y, z) = \frac{\hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z}{(x^2 + y^2 + z^2)^{3/2}} = \frac{1}{r^2}\hat{r}$$

where the second form just uses the definition of the position vector $\vec{r} = \hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z$ and the radial unit vector $\hat{r} = \vec{r}/r$. Use your result for the divergence to test Gauss' Theorem for the spherical volume V bounded by the spherical surface S with radius R, centered on the origin. The surface integral is very easy to calculate since $r = R$ on this surface, and the unit vector $\hat{r}$ is perpendicular to the surface everywhere. However, the result will look like a violation of Gauss' Theorem. Can you see what is the source of the problem? (That's a pun, by the way.) You might recognize that $\vec{E}$ is proportional to the electric field from a point charge.

(1) Since $\vec{\nabla} \cdot \vec{A} = \vec{C} \cdot \vec{\nabla} f$ (which is simple to prove just by looking at the components),

$$\int_V \vec{\nabla} \cdot \vec{A}\, dV = \vec{C} \cdot \int_V \vec{\nabla} f\, dV = \oint_S \vec{A} \cdot d\vec{S} = \vec{C} \cdot \oint_S f\, d\vec{S}$$

From here, you can write that

$$\vec{C} \cdot \left[\int_V \vec{\nabla} f\, dV - \oint_S f\, d\vec{S} \right] = 0$$

and argue that since $\vec{C}$ is arbitrary, then the expression in square brackets must be zero. Or you could go through this equation component by component, setting $\vec{C} = \hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

(2) Three sides of the surface integral give $0 \times L^2$ and the other three sides give $L \times L^2$ so the surface integral is $3L^3$. The volume integral is of $\vec{\nabla} \cdot A = 3$ over the volume L^3 so the volume integral is $3L^3$. These are equal. The theorem works.

(3) Label the sides of the square 1, 2, 3, and 4, starting with the side on the x-axis and counting counter clockwise. The vector field along these sides are $\vec{A}_1 = \hat{\jmath}x$, $\vec{A}_2 = -\hat{\imath}y + \hat{\jmath}L$, $\vec{A}_3 = -\hat{\imath}L + \hat{\jmath}x$, and $\vec{A}_4 = -\hat{\imath}y$, so $\vec{A}_1 \cdot d\vec{\ell} = 0$, $\vec{A}_2 \cdot d\vec{\ell} = L^2$, $\vec{A}_3 \cdot d\vec{\ell} = L^2$, and $\vec{A}_4 \cdot d\vec{\ell} = 0$. Therefore, the line integral is $2L^2$. Since $\vec{\nabla} \times \vec{A} = \hat{k}[1 - (-1)] = 2\hat{k}$ and the surface normal vector is $\hat{k}$, the surface integral is $2L^2$. These are equal. The theorem works.

(4) The divergence of this field is zero, although you can't apply that to the origin since the field is infinite there. The surface integral is simply

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{R^2} \oint \hat{r} \cdot \hat{r}\, dS = \frac{1}{R^2} \oint dS = \frac{1}{R^2} 4\pi R^2 = 4\pi$$

which is not zero. In fact $\vec{\nabla} \cdot \vec{E}$ is a δ-function at the origin. (More on this later.)

PHYS2063 Wave Physics Homework #14 Due Thursday 13 Oct 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

A long straight thin wire runs along the z-axis and carries a charge λ per unit length. Find the magnitude and direction of the electric field $\vec{E}$ at a distance $r = \sqrt{x^2 + y^2}$ from the wire. Take advantage of symmetries to identify the easiest "Gaussian surface" S, and then apply Gauss' Law in integral form. Express your answer for $\vec{E}(x, y)$ in terms of x, y, and their unit vectors.

Then take the divergence $\vec{\nabla} \cdot \vec{E}(x, y)$ of your result and show that you get the answer you expect, at least so long as you stay away from the z-axis.

The electric field has no choice but to point radially away from the wire, so choose a Gaussian cylinder of radius r and length L. Therefore

$$\oint_S \vec{E}\cdot d\vec{S} = E\times 2\pi r\times L = 4\pi\,\lambda L \qquad \text{so} \qquad E = \frac{1}{2}\frac{\lambda}{r}$$

The radial unit vector is $\hat{r} = \vec{r}/r = (\hat{\imath}x + \hat{\jmath}y)/\sqrt{x^2+y^2}$, so

$$\vec{E}(x,y) = \frac{1}{2}\frac{\lambda}{r}\frac{\hat{\imath}x+\hat{\jmath}y}{\sqrt{x^2+y^2}} = \frac{\lambda}{2}\frac{\hat{\imath}x+\hat{\jmath}y}{x^2+y^2}$$

The divergence of this field is

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \frac{\lambda}{2}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]\\
&= \frac{\lambda}{2}\left[\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}+\frac{1}{x^2+y^2}-\frac{2y^2}{(x^2+y^2)^2}\right]\\
&= \frac{\lambda}{2}\left[\frac{2}{x^2+y^2}-\frac{2(x^2+y^2)}{(x^2+y^2)^2}\right] = \frac{\lambda}{2}\left[\frac{2}{x^2+y^2}-\frac{2}{x^2+y^2}\right] = 0
\end{aligned}$$

except along the line where $x = y = 0$, that is, the z-axis. This is all correct, since there is no charge off the z-axis, so the divergence ought to be zero there.

PHYS2063 Wave Physics Homework #15 Due Tuesday 18 Oct 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

In class, we derived the electric and magnetic fields, as a function of time, for a plane wave linearly polarized with its electric field in the x-direction and moving in the z-direction. This homework assignment concerns two variations of this solution.

(1) Show that the electric field, where $\omega = kc$,

$$\vec{E}(z,t) = \hat{\imath}E_0\cos(kz - \omega t) + \hat{\jmath}E_0\sin(kz - \omega t)$$

also solves the electromagnetic wave equation for $\vec{E}(x,y,z,t)$. Find the magnetic field $\vec{B}(x,y,z,t)$, and demonstrate that $\vec{E} \times \vec{B}$ is in the right direction. Describe the behavior of the polarization of $\vec{E}(z,t)$, and come up with a name for it.

(2) Find a solution $\vec{E}(x,y,z,t)$ to the electric field wave equation for a plane wave linearly polarized with its electric field in the x-direction, but this time moving in the direction of the line $y = z$ (with y and z both increasing) in the yz plane. Find the magnetic field $\vec{B}(x,y,z,t)$, and demonstrate that $\vec{E} \times \vec{B}$ is in the right direction.

(1) Since both $E_x(z,t)$ and $E_y(z,t)$ are of the form $f(z-ct)$, it is obvious that this electric field solves the wave equation for a wave moving towards $+z$. For the magnetic field, use

$$\begin{aligned}
\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E} &= c\hat{\imath}\,\frac{\partial E_y}{\partial z} - c\hat{\jmath}\,\frac{\partial E_x}{\partial z} = kcE_0\left[\hat{\imath}\,\cos(kz-\omega t) + \hat{\jmath}\,\sin(kz-\omega t)\right] \\
\text{so} \qquad \vec{B} &= -\hat{\imath}\,E_0\sin(kz-\omega t) + \hat{\jmath}\,E_0\cos(kz-\omega t)
\end{aligned}$$

Notice that $|\vec{B}| = |\vec{E}||$ as it should be. (Also notice that $\vec{E}$ and $\vec{B}$ are orthogonal, as they should be.) The cross product is

$$\vec{E}\times\vec{B} = \hat{k}\,E_0^2\cos^2(kz-\omega t) + \hat{k}\,E_0^2\sin^2(kz-\omega t) = \hat{k}\,E_0^2$$

which points in the $+z$ direction, as it should. As for the polarization, consider the direction of $\vec{E}$ as a function of time for a fixed z, say $z=0$. The vector rotates in a circle with angular frequency ω, keeping the same magnitude. We call this "circular polarization."

(2) Linearly polarized in the x-direction means that $\vec{E} = \hat{\imath}\,E_0 f(x,y,z,t)$. Moving along the line $y=z$ in the positive direction means that the unit vector in the direction of the wave is $(\hat{\imath}+\hat{k})/\sqrt{2}$, so we expect that $f(x,y,z,t) = g(y/\sqrt{2}+z/\sqrt{2}-ct)$, and in fact

$$\frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial y^2} + \frac{\partial^2\vec{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \hat{\imath}\,\left[0+\frac{1}{2}+\frac{1}{2}-\frac{1}{c^2}c^2\right]E_0 g' = 0$$

showing that this is indeed a solution to the wave equation. For the magnetic field, use

$$\begin{aligned}
\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E} &= -c\hat{\jmath}\,\frac{\partial E_x}{\partial z} + c\hat{k}\,\frac{\partial E_x}{\partial y} = -cE_0\frac{1}{\sqrt{2}}g'\left(\frac{y}{\sqrt{2}}+\frac{z}{\sqrt{2}}-ct\right)(\hat{\jmath}-\hat{k}) \\
\text{so} \qquad \vec{B} &= E_0\frac{1}{\sqrt{2}}g\left(\frac{y}{\sqrt{2}}+\frac{z}{\sqrt{2}}-ct\right)(\hat{\jmath}-\hat{k})
\end{aligned}$$

(Also notice that $\vec{E}$ and $\vec{B}$ are orthogonal, as they should be.) The cross product is proportional to

$$\hat{\imath}\times(\hat{\jmath}-\hat{k}) = \hat{k}+\hat{\jmath}$$

which is indeed in the direction of propagation.

PHYS2063 Wave Physics Homework #16 Due Thursday 20 Oct 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

An electromagnetic plane wave with linear polarization propagates with an electric field

$$\vec{E}(x, y, z, t) = \hat{\epsilon} E_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$$

where $\omega = |\vec{k}|c$. Assume the wave propagates away from the origin in the direction $x = y = z$ in the first octant, i.e. x, y, and z all positive, and that the polarization unit vector $\hat{\epsilon}$ lies in the xy plane.

Find the wave vector $\vec{k}$ in terms of $k = |\vec{k}|$ and unit vectors in the x, y, and z directions.

Find the polarization unit vector $\hat{\epsilon}$.

Find the magnetic field $\vec{B}(x, y, z, t)$ in terms of E_0, k, ω, and unit vectors in the x, y, and z directions.

Find the Poynting vector $\vec{S}(x, y, z, t)$ in terms of E_0, k, ω, and unit vectors in the x, y, and z directions.

I encourage you to solve this problem using results that we derived in class, although you are welcome to find the solution by going back to Maxwell's Equations if you prefer.

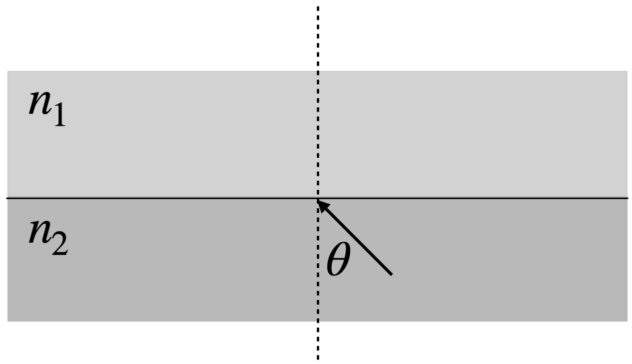
$$
\begin{aligned}
\vec{k} &= \frac{k}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right) \\
\hat{\epsilon} &= a\hat{i}+b\hat{j} \qquad \text{where} \qquad a^2+b^2=1 \\
\vec{k}\cdot\hat{\epsilon} &= \frac{k}{\sqrt{3}}(a+b)=0 \qquad \text{so} \quad b=-a \\
\hat{\epsilon} &= \frac{1}{\sqrt{2}}\left(\hat{i}-\hat{j}\right) \\
\vec{B} &= \vec{k}\times\vec{E} \\
\text{so calculate} \qquad \vec{k}\times\hat{\epsilon} &= \frac{k}{\sqrt{6}}\left(-\hat{k}-\hat{k}+\hat{j}+\hat{i}\right)=\frac{k}{\sqrt{6}}\left(\hat{i}+\hat{j}-2\hat{k}\right) \\
\vec{S} &= \frac{1}{4\pi}\vec{E}\times\vec{B}=\frac{1}{4\pi}\frac{\vec{k}}{k}E_0^2\cos^2(\vec{k}\cdot\vec{r}-\omega t)
\end{aligned}
$$

where I write the unit vector in the propagation direction as $\vec{k}/k$ to avoid confusion with the unit vector in the z-direction.

PHYS2063 Wave Physics Homework #17 Due Tuesday 25 Oct 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

(1) Light impinges on the interface between two materials as shown below:



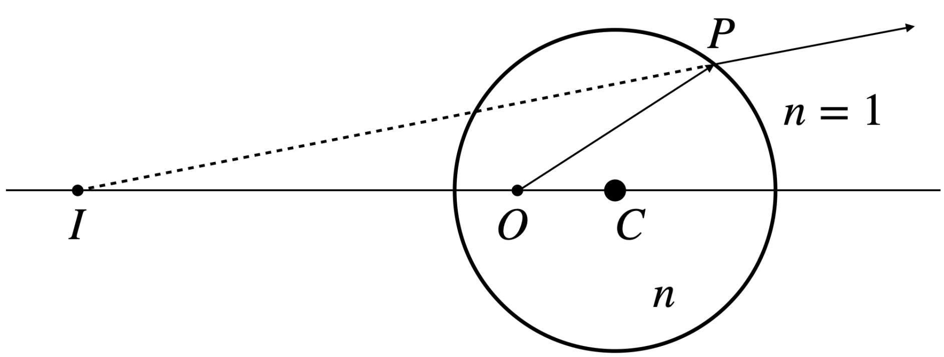
Under the condition that $n_2 > n_1$, find an angle θ for which the light cannot propagate into the top medium. (This phenomenon is called "total internal reflection.")

(2) (Pain Problem 11.2) A parallel plate of glass of thickness d has a non-uniform refractive index n given by

$$n = n_0 - \alpha r^2$$

where n_0 and α are constants and r is the distance from a certain line perpendicular to the sides of the plate. Show that this plate behaves as a converging lens of focal length $1/2\alpha d$. You should assume the same small angle "thin lens" approximation we used in class.

(3) (Pain Problem 11.8) An object O is imbedded inside a glass sphere of radius R and index of refraction n, as shown in the figure.



The object is located at a distance R/n from the center of the sphere. Outside the sphere is a vacuum ($n = 1$). Show that any ray OP, when projected back to the line connecting the object with the center of the sphere, meets at the distance $IC = nR$. (This is, apparently, the principle used in the "oil immersion microscope."

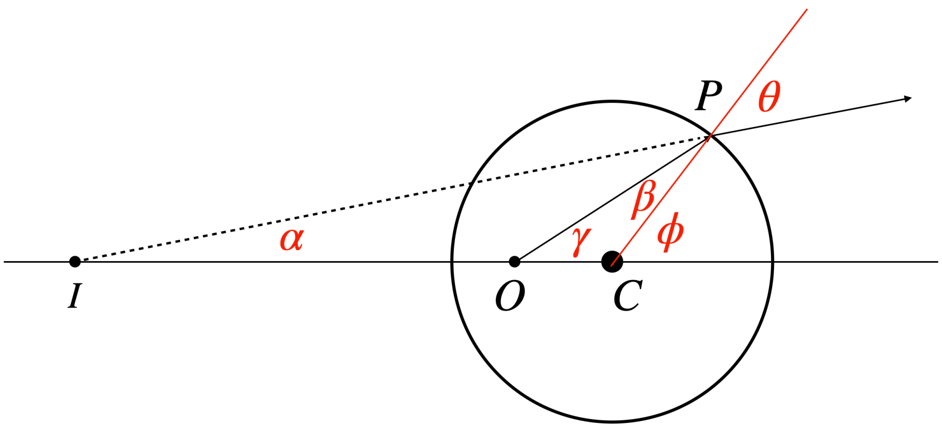
(1) From $n_1 \sin\theta_1 = n_2 \sin\theta_2$ we have $\sin\theta_1 = (n_2/n_1)\sin\theta$. Since $n_2/n_1 > 1$ and $\sin\theta_1 \le 1$, we reach a critical angle when $\theta = \sin^{-1}(n_1/n_2)$. For angles larger than this, there can be no propagation into medium #1.

(2) If $z = z(r)$ is the distance past the plate to get an equal-time wave front, then

$$\frac{d}{c/n_0} = \frac{d}{c/n} + \frac{z}{c} \qquad \text{so} \qquad n_0 d = nd + z$$

which reduces to $z = \alpha r^2 d$ for the given form for $n(r)$. The ray intersects the axis at a distance f at an angle $\theta = \tan^{-1}(r/f) \approx r/f$. As we discussed in class, we also have $z = f - f\cos\theta \approx f\theta^2/2 = r^2/2f$. Therefore $f = r^2/2z = r^2/2\alpha r^2 d = 1/2\alpha d$.

(3) We use the law of sines with the triangles IPC and OPC below:



We have $n\sin\beta = \sin\theta$. Note that $\alpha + \theta + (180^\circ - \phi) = 180^\circ$, so $\alpha = \phi - \theta$. Similarly $\gamma = \phi - \beta$. Applying the Law of Sines to OPC, where R is the radius of the sphere, we have

$$\frac{R}{\sin\gamma} = \frac{R/n}{\sin\beta} = \frac{R}{\sin\theta} \qquad \text{so} \qquad \theta = \gamma = \phi - \beta \qquad \text{and} \qquad \alpha = \phi - (\phi - \beta) = \beta$$

Applying the Law or Sines to IPC we have

$$\frac{R}{\sin\alpha} = \frac{IC}{\sin\theta} \qquad \text{so} \qquad \frac{IC}{R} = \frac{\sin\theta}{\sin\alpha} = \frac{\sin\theta}{\sin\beta} = n$$

Therefore $IC = nR$.

PHYS2063 Wave Physics Homework #18 Due Tuesday 1 Nov 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

A beam of helium atoms emerge from a double slit mask and impinge on a screen 1950 mm away. The spacing between the slits is 8 μm. For the atoms which take 800 μs to reach the screen, how far apart are the interference maxima? You can check your answer against the figure we showed in class and the paper from which it was taken, both of which have links on the course web page.

Homework #18 <u>Solutions</u>

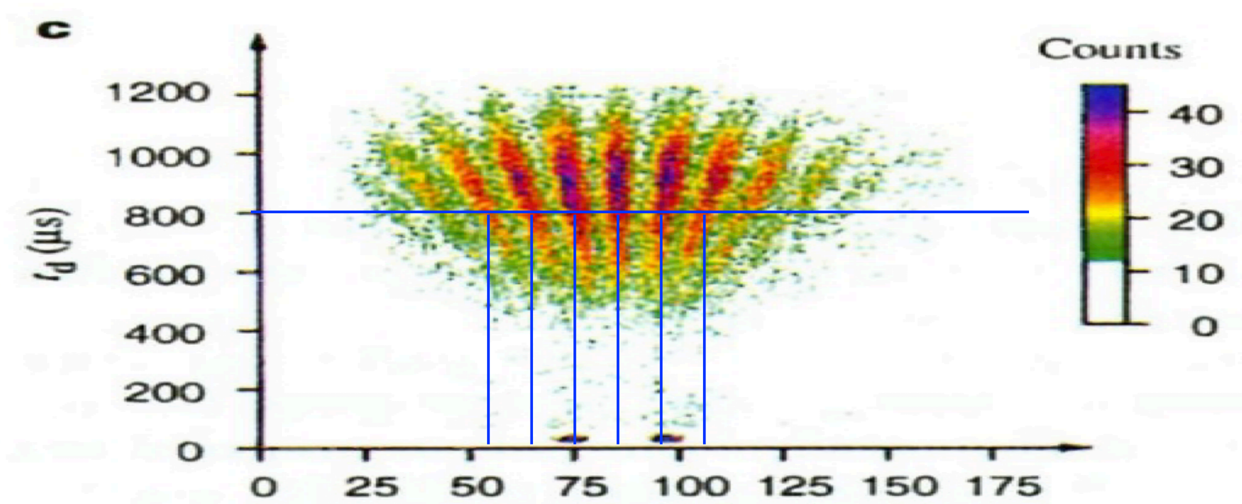
The velocity of the helium atoms is $1.95/8 \times 10^{-4} = 2.4 \times 10^3$ m/s.

The mass of a helium atom is 6.6×10^{-27} kg, so the momentum is 1.6×10^{-3} kg·m/s.

Planck's constant $h = 6.6 \times 10^{-34}$ kg·m^2/s, so its wavelength $\lambda = h/p = 4.2 \times 10^{-11}$ m.

The angle between maxima is $\sin\theta = \lambda/d = 5.2 \times 10^{-5}$.

The separation between maxima is therefore $1.95 \sin\theta = 1.0 \times 10^{-5}$ m=10 μm. This looks to be in good agreement with the figure below, although I can't find precise confirmation that the horizontal scale is in microns.



PHYS2063 Wave Physics Homework #19 Due Tuesday 8 Nov 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

(1) An ice pick is a device the size of a screwdriver with a sharp point connected to a handle:

Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations which do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it *in seconds*.

(2) This question is a bit open ended. I want you to do some investigating about a fascinating phenomenon that has to do with the the wave nature of matter.

Helium turns into a liquid at atmospheric pressure at a temperature of about four degrees above absolute zero. Calculate the deBroglie wavelength of a helium atom at a temperature $T = 2K$. You can assume the energy of the atom is given by the thermal energy $E = (3/2)kT$, where k is Boltzmann's constant. Compare this wavelength to the size of helium atom. What does this suggest to you about the behavior of liquid helium at this very low temperature? Identify the phenomenon to which this corresponds.

You might enjoy the video http://www.alfredleitner.com/p/liquid-helium.html.

(1) We're looking for a "rough order of magnitude" estimate, so go crazy with the approximations. Model the ice pick as a mass m and length L, standing vertically on the point, i.e. and inverted pendulum. The angular acceleration is $\ddot{\theta}$, the moment of inertia is mL^2 and the torque is $mgL\sin\theta$ where θ is the angle from the vertical. So $mL^2\ddot{\theta} = mgL\sin\theta$ or $\ddot{\theta} = \sqrt{g/L}\sin\theta$. Since $\theta \ll 0$ as the pick starts to fall, take $\sin\theta = \theta$ so

$$\begin{aligned}
\theta(t) &= A\exp\left(\sqrt{\frac{g}{L}}t\right) + B\exp\left(-\sqrt{\frac{g}{L}}t\right) \\
x_0 \equiv \theta(0)L &= (A+B)L \\
p_0 \equiv m\dot{\theta}(0)L &= m\sqrt{\frac{g}{L}}(A-B)L = \sqrt{m^2gL}(A-B)
\end{aligned}$$

Let the uncertainty principle relate x_0 and p_0, i.e. $x_0p_0 = \sqrt{m^2gL^3}(A^2-B^2) = \hbar$. Now ignore B; the exponential decay will become irrelevant quickly. You can notice that the pick is falling when it is tilting by something like $1° = \pi/180$, so solve for a time T where $\theta(T) = \pi/180$. Then

$$T = \sqrt{\frac{L}{g}}\ln\frac{\pi/180}{A} = \sqrt{\frac{L}{g}}\left(\frac{1}{4}\ln\frac{m^2gL^3}{\hbar^2} - \ln\frac{180}{\pi}\right)$$

Take $L = 10$ cm, so $\sqrt{L/g} \approx 0.1$ sec, but the action is in the logarithms. (It is worth your time to confirm that the argument of the logarithm in the first term is indeed dimensionless.) Now $\ln(180/\pi) \approx 4$ but the first term appears to be much larger. This is good, since it means that quantum mechanics is driving the result. For $m = 0.1$ kg, find $m^2gL^3/\hbar^2 = 10^{64}$, and so $T = 0.1 \text{ sec} \times (147/4 - 4) \sim 3$ sec. I'd say that's a surprising and interesting result.

(2) First calculate the deBroglie wavelength. I will use SI units everywhere.

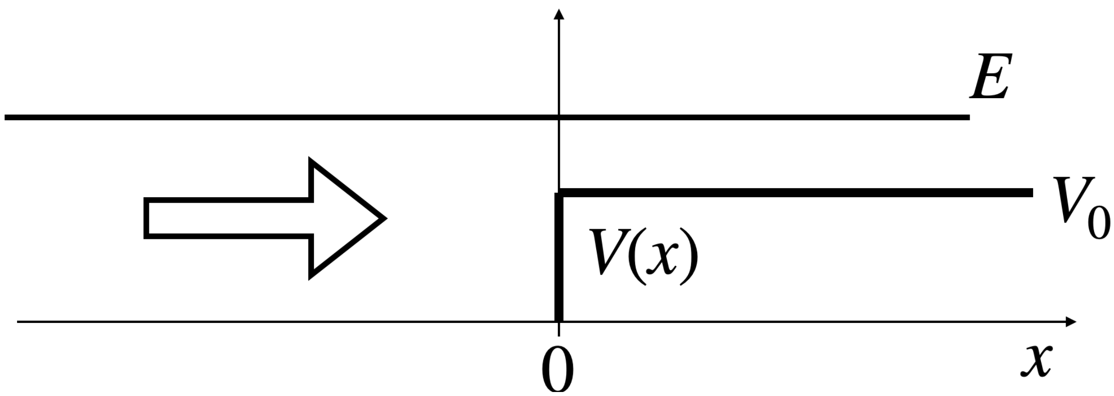
$$\begin{aligned}
E &= \frac{3}{2}kT = \frac{3}{2} \times (1.38\times 10^{-23}) \times 2 = 4.14 \times 10^{-23} \text{ Joules} \\
p &= \sqrt{2mE} = \sqrt{2 \times (6.64\times 10^{-27})(4.14\times 10^{-23})} = 7.41 \times 10^{-25} \text{ kg}\cdot\text{m/sec} \\
\lambda &= \frac{h}{p} = 6.63\times 10^{-34}/7.41\times 10^{-25} = 8.94\times 10^{-10} \text{ m} = 8.94 \text{ Å}
\end{aligned}$$

A quick Google search tells you that the size of a helium atom is about 0.1 nm=1 Å, or maybe as large as 2 Å if you include some extent of the electron orbitals. The point is that at $T = 2K$, the wavelength is large enough to include several atoms. This suggests that liquid helium should be have, somehow, quantum mechanically at this temperature. Indeed, at $T = 2.17$ K, liquid helium undergoes a phase transition to a "super fluid" state.

PHYS2063 Wave Physics Homework #20 Due Thursday 10 Nov 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

Finish the problem we started in class, namely that of a particle wave incident from the left on a square potential step of height V_0 at $x = 0$ and with energy $E > V_0$:



That is, find the reflection and transmission coefficients R and T as a function of E/V_0 and show that $R + T = 1$. Make plots of R and T as a function of E/V_0. Comment on the behavior as $E/V_0 \to \infty$ and $E \to V_0$.

Write down the form of the two solutions for $x < 0$ and $x > 0$.

$$u_I(x) = Ae^{ik_I x} + Be^{-ik_I x} \qquad \frac{\hbar^2 k_I^2}{2m} = E \qquad u_{II}(x) = Ce^{ik_{II} x} \qquad \frac{\hbar^2 k_{II}^2}{2m} = E - V_0$$

Match the solutions and their derivatives at $x = 0$.

$$A + B = C \qquad k_I A - k_I B = k_{II} C = k_{II}(A + B)$$

Now solve for the ratios of the reflected and transmitted coefficients to the incident coefficient.

$$\frac{B}{A} = \frac{k_I - k_{II}}{k_I + k_{II}} = \frac{1 - k_{II}/k_I}{1 + k_{II}/k_I} \qquad \text{and} \qquad \frac{C}{A} = 1 + \frac{B}{A} = \frac{2k_I}{k_I + k_{II}} = \frac{2}{1 + k_{II}/k_I}$$

To get the reflection and transmission coefficients, we need to remember that the flux is proportional to k, so

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{1 - k_{II}/k_I}{1 + k_{II}/k_I}\right)^2 \qquad \text{and} \qquad T = \frac{k_{II}}{k_I}\left|\frac{C}{A}\right|^2 = \frac{4k_{II}/k_I}{(1 + k_{II}/k_I)^2}$$

It is easy to show that the sum is unity. If we write $\alpha = k_{II}/k_I$, then

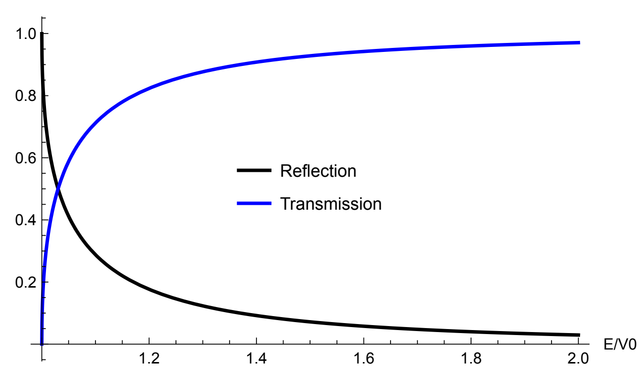
$$R + T = \frac{(1-\alpha)^2}{(1+\alpha)^2} + \frac{4\alpha}{(1+\alpha)^2} = \frac{1 + 2\alpha + \alpha^2}{(1+\alpha)^2} = 1$$

From this you can see immediately that since $k_{II} \to k_I$ as $E \to \infty$, you expect that $R \to 0$ and $T \to 1$ in this limit. Furthermore, as $E \to V_0$, $k_{II} \to 0$ so $R \to 1$ and $T \to 0$.

To plot the expressions for R and T, it is simplest to express the answers in terms of

$$\frac{k_{II}}{k_I} = \sqrt{\frac{E - V_0}{E}} = \sqrt{1 - \frac{1}{\varepsilon}} \qquad \text{where} \qquad \varepsilon \equiv \frac{E}{V_0}$$

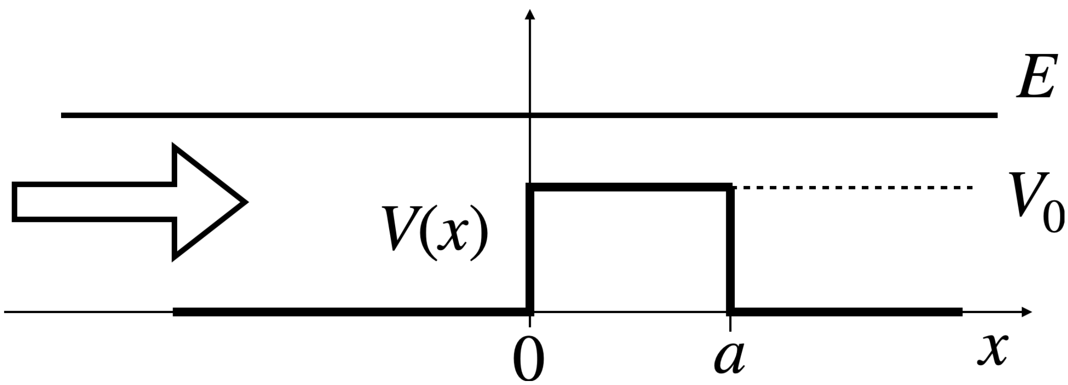
The expressions themselves are a bit messy, so I won't write them out. The plot below comes from Mathematica.



PHYS2063 Wave Physics Homework #21 Due Tuesday 15 Nov 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

In class we analyzed quantum mechanical penetration through a square barrier, that is, for the energy E less than the barrier height V_0. Now analyze the same problem, but for $E > V_0$:

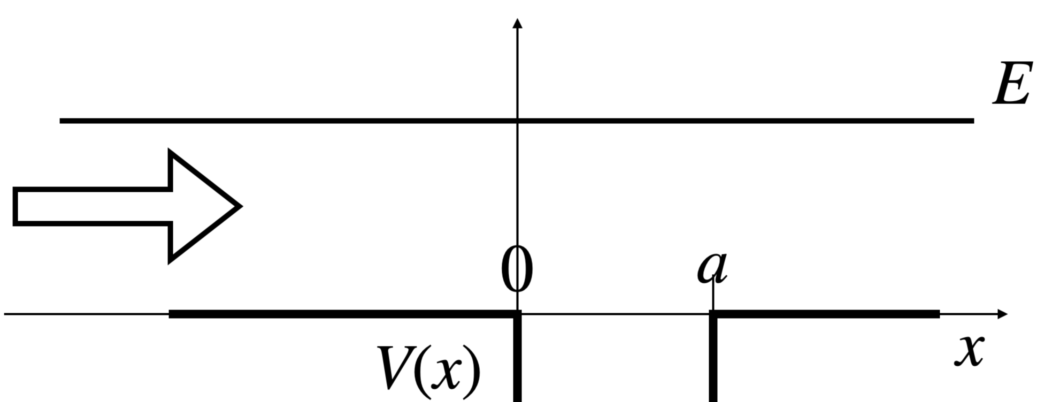


Find the transmission coefficient T and plot it as a function of E/V_0. I showed you how to make the plot for barrier penetration using

$$V_0 = g\,\frac{\hbar^2}{2ma^2} \qquad \text{where} \qquad g = 16$$

but you can use a different value for g if you want. In any case, also plot the result for $E < V_0$ and show that the two curves "connect."

Also analyze the case of transmission past a "quantum well" of the form



and plot the transmission coefficient as a function of energy $E > 0$.

In both cases, there are values of the energy for which there is perfect transmission, that is, $T = 1$. What is the physical significance of these energies? That is, what is the property of the particle wave that leads to perfect transmission?

We follow from the result in class for C/A but put $q \to iq$. That is

$$\frac{C}{A} = \frac{-4kq\, e^{-i(k-q)a}}{(e^{2iqa}-1)(k^2+q^2) - 2kq(e^{2iqa}+1)} = \frac{-2kq\, e^{-i(k-q)a}}{i\sin(qa)(k^2+q^2) - 2kq\cos(qa)}$$

with $\hbar^2 q^2/2m = E - V_0$, as well as $\hbar^2 k^2/2m = E$. Therefore

$$\frac{1}{T} = \frac{1}{4k^2q^2}\left[\sin^2(qa)(k^2+q^2)^2 + 4k^2q^2\cos^2(qa)\right] = 1 + \frac{(k^2-q^2)^2}{4k^2q^2}\sin^2(qa)$$

and $(k^2-q^2)^2/4k^2q^2 = V_0^2/E(E-V_0)$. It's interesting that this is the expression for barrier penetration but with $q \to iq$, but I'm not sure that isn't an accident.

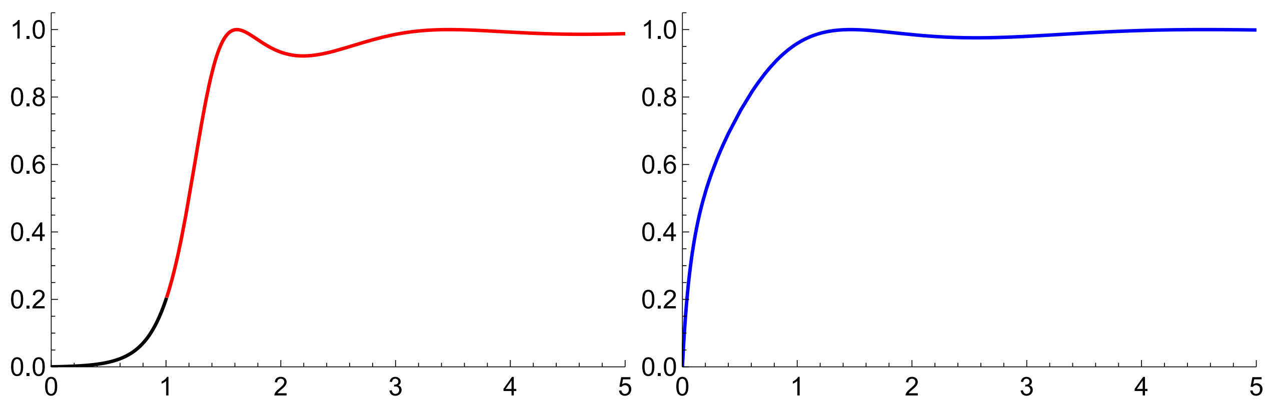
For the case of a potential well, you just change $V_0 \to -V_0$.

The transmission coefficient becomes unity whenever $qa = n \times 2\pi$ for some integer n, that is

$$\lambda = \frac{h}{p} = \frac{h}{\hbar q} = \frac{2\pi}{n \times 2\pi/a} = \frac{a}{n} \qquad \text{or} \qquad n\lambda = a$$

So there is perfect transmission if an integral number of wavelengths span the barrier or well.

The MATHEMATICA notebook makes the plots below, barrier on the left and well on the right:

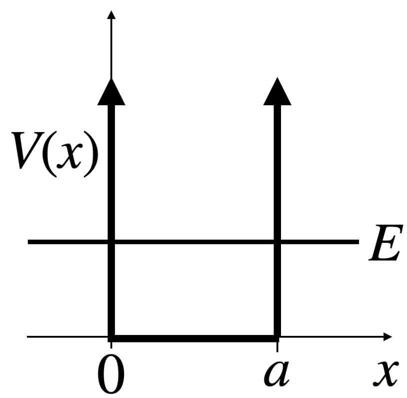


The barrier plot includes the barrier penetration result from class, showing that it joins smoothly onto the case for $E > V_0$.

PHYS2063 Wave Physics Homework #22 Due Thursday 17 Nov 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

In class we derived the wave functions and energy levels for the infinite square well in one dimension with width a, where the well was placed symmetrically between $x = -a/2$ and $+a/2$. Show that you get the same solution, for the normalized wave functions as well as the energy levels, if you instead place one side of the well at $x = 0$, namely



You will find that the solution follows very closely from our work from several weeks ago on standing waves on a stretched string.

Define k through $E \equiv \hbar^2 k^2/2m$. Then the time-independent Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} = Eu = \frac{\hbar^2k^2}{2m}u \qquad \text{so} \qquad \frac{d^2u}{dx^2} = -k^2u$$

so $u(x) = A\cos(kx) + B\sin(kx)$. Requiring $u(0) = A = 0$ leaves us with $u(x) = B\sin(kx)$. In order to satisfy $u(a) = B\sin(ka) = 0$ we must have $ka = n\pi$ for positive integers n. The energy levels are therefore

$$E = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2\pi^2}{2ma^2}n^2$$

which is what we found in class for the symmetric well. The normalization follows from

$$\int_0^a B^2\sin^2\left(\frac{n\pi x}{a}\right)dx = \frac{B^2}{4}a\left[2 - \frac{\sin(2\pi n)}{\pi n}\right]_0^a = \frac{B^2a}{2} = 1 \qquad \text{so} \qquad B = \sqrt{\frac{2}{a}}$$

The wave functions then become

$$u(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

If we switch to a variable $\xi \equiv x - a/2$ then

$$\begin{aligned} u(\xi) &= \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi\xi}{a} + \frac{n\pi}{2}\right) \\ &= \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi\xi}{a}\right) \qquad n\,\text{odd} \\ &= \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi\xi}{a}\right) \qquad n\,\text{even} \end{aligned}$$

which is just what we got in class, ignoring overall signs (which just serves to redefine the normalization constant).

PHYS2063 Wave Physics Homework #23 Due Tuesday 29 Nov 2022

This homework assignment is due at the start of class on the date shown. You may submit a PDF of your solutions to the Canvas page for the course, or bring a paper copy to class.

(1) In class we studied the expectation value of position with an infinite-square-well wave function $u(x)$ that was equal parts $u_1(x)$ and $u_2(x)$ with real coefficients. Recalculate this expectation value with a similar wave function, but which has an arbitrary phase between $u_1(x)$ and $u_2(x)$, namely

$$u(x) = \frac{1}{\sqrt{2}}\left[u_1(x) + e^{i\phi}u_2(x)\right]$$

Confirm that you get the same result we had in class for $\phi = 0$. You should find the integral we did in class useful.

(2) Now calculate the ime dependent expectation value of position for an equal superposition of $\psi_1(x,t) = u_1(x)e^{-iE_1t/\hbar}$ and $\psi_2(x,t) = u_2(x)e^{-iE_2t/\hbar}$. You should find the results of Problem (1) useful. How would you describe the behavior of the particle in the well, briefly?

(3) In class we worked through the numerical calculation of the finite square well with

$$V_0 = g\frac{\hbar^2}{2ma^2}$$

with a "very deep well" namely $g = 100$. Repeat the calculation for an extremely deep well with $g = 1000$, and compare the bound state energies with the results for an infinitely deep well. You are welcome to adapt the MATHEMATICA notebook we used in class.

Homework #23 <u>Solutions</u>

(1) The integrand for the calculation of the expectation value is

$$u^*(x)xu(x) = \frac{1}{2}x\,u_1^2(x) + \frac{1}{2}x\,u_2^2(x) + \cos\phi\,x\,u_1(x)u_2(x)$$

Integrating each of the first two terms gives zero because they are odd functions integrated over $-a/2 \le x \le +a/2$. Integrating the third term is just $\cos\phi$ times the integral we did in class. Therefore

$$\langle x\rangle = \int_{-a/2}^{a/2} u^*(x)xu(x)\,dx = \cos\phi\,\frac{16a}{9\pi^2}$$

Clearly we get the same answer as in class when $\phi = 0$, but it is interesting to note that the result can be zero or negative depending on the value of ϕ.

(2) The calculation for the expectation value now uses the wave function

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left[u_1(x)e^{-iE_1t/\hbar} + u_2(x)e^{-iE_2t/\hbar}\right] = e^{-iE_1t/\hbar}\frac{1}{\sqrt{2}}\left[u_1(x) + u_2(x)e^{-i(E_2-E_1)t/\hbar}\right]$$

The overall factor of $e^{-iE_1t/\hbar}$ cancels in $\psi^*\psi$, so this reduces to Problem (1) with $\phi = -\omega_{21}t$ where $\omega_{21} \equiv (E_2 - E_1)t/\hbar$. So the expectation value is

$$\langle x\rangle = \cos(\omega_{21}t)\,\frac{16a}{9\pi^2}$$

The probability density "sloshes" left and right with frequency ω_{21}. It represents the particle "in motion."

(3) See the Mathematica notebook. The following plot compares the values $g\epsilon$ for the different solutions to the values of π^2n^2 as a function of n. The comparison is very good, especially at the lower values, as expected.