PHYS 2100 Midterm Exam #1

Introduction to Methods of Theoretical Physics

Fall 1999 NAME: _____Solutions___

You have two hours to complete this exam. There are a total of four problems and you are to solve all of them. *Not all the problems are worth the same number of points.*

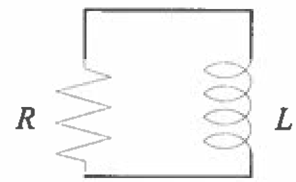
You may use your textbooks and class notes and handouts, or other books. You *may not* share these resources with another student during the test. For reasons of equity, nobody may use laptop computers.

Indicate any figures or tables you use in your calculations. Show all work!

GOOD LUCK!

Problem	Score	Worth
1.	________	20
2.	________	20
3.	________	30
4.	________	30
Total Score:	__________	100

Problem 1 (10+5+5=20 points): A circuit is constructed from a resistor R and an inductor L:



a) Write the differential equation that must be solved to find the current $i(t)$ as a function of time.

The sum of the voltage drops around the closed loop must be zero. Therefore

$$V_R + V_L = 0$$

$$iR + L\frac{di}{dt} = 0$$

b) Show that the magnetic energy, stored in the inductor, disappears at a rate equal to the power dissipated by the resistor.

$$\frac{d(E_L)}{dt} = \frac{d\left(\frac{1}{2}Li^2\right)}{dt} = Li\frac{di}{dt} = i(-iR) = -i^2R$$

c) The circuit is set up initially with a steady current $i(0) = i_0$. Show that $i(t) = i_0e^{-t/\tau}$. Find an expression for τ in terms of L and R and explain why τ is a "natural" time scale.

Substitute into the differential equation in part (a) to get

$$i_0e^{-t/\tau}R + L\left(-\frac{1}{\tau}\right)(i_0e^{-t/\tau}) = \left(R - \frac{L}{\tau}\right)(i_0e^{-t/\tau}) = 0$$

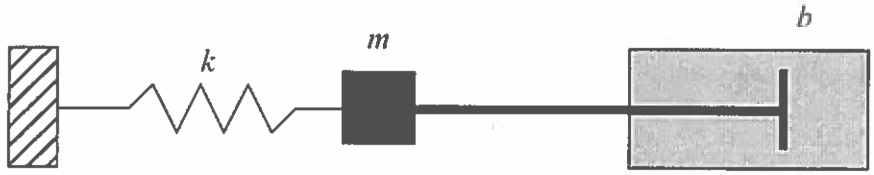
for

$$\tau = \frac{L}{R}$$

This is the "natural" time scale since it is the characteristic time in which the current decays away. It is also the only time scale that you can make out of the available parameters.

Problem 2 (10+10=20 points). A particular oscillator as a quality factor $Q = 150$ and a resonant frequency very close to 10 Hz (i.e. 10/sec).

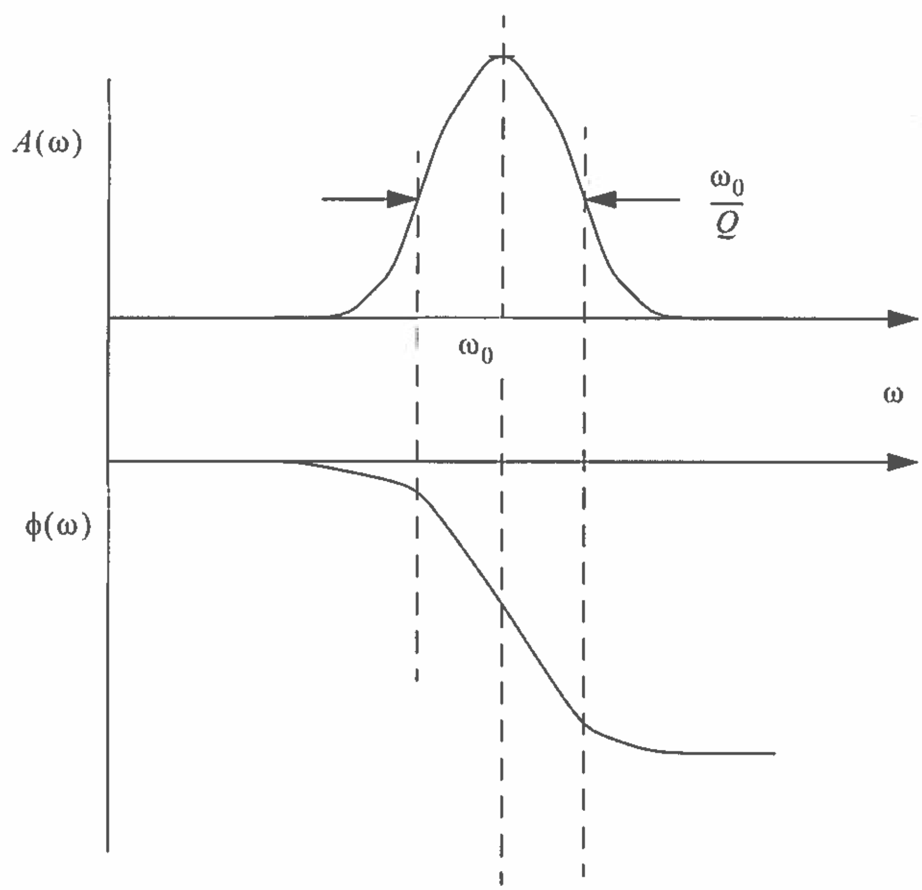
a) Draw and label a picture of a mechanical oscillator with one degree of freedom that satisfies these conditions. Be sure to assign appropriate numerical values to all necessary parameters.



$$Q = \omega_0 \frac{b}{m} = 150 \text{ and } \omega_0 = (k/m)^{1/2} = 10/\text{sec}$$

so pick $m = 15$ kg, which implies $b = 1$ kg/sec and $k = 1500$ N/m.

b) The oscillator is forced according to the real part of $fe^{i\omega t}$ and it responds according to the real part of $x(t) = Ae^{i(\omega t + \phi)}$, where f, A, ω, and ϕ are real numbers. Sketch the functional forms of $A(\omega)$ and $\phi(\omega)$. Be as accurate as the given information allows.



Problem 3 (5+5+10+10=30 points): A string of length L, linear mass density ρ, and under tension T, vibrates between two fixed ends. The shape is given by $y(x, t)$, where t is time and x measures the horizontal position along the string. Assume the small angle approximation.

a) Write an expression for the kinetic energy ΔK of the piece of string between x and $x + \Delta x$.

$$\Delta K = \frac{1}{2}(\Delta m)v^2 = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 \Delta x$$

b) Define a potential energy ΔU by the work done by tension in bending and stretching the flat piece of string between x and $x + \Delta x$ to a length Δs. That is, $\Delta U = T(\Delta s - \Delta x)$.

$$\Delta U = T(\Delta s - \Delta x) = T([\Delta x^2 + \Delta y^2]^{1/2} - \Delta x) = T\left(\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]^{1/2} - 1\right)\Delta x$$

$$\Delta U \approx T\left(\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right] - 1\right)\Delta x = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \Delta x$$

c) For an arbitrary traveling wave with amplitude A, show that the energy $\Delta E = \Delta K + \Delta U$ travels with the same speed as the wave but with an amplitude proportional to A^2.

$$\Delta E = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 \Delta x + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \Delta x = \frac{1}{2}\rho\left[\left(\frac{\partial y}{\partial t}\right)^2 + s^2\left(\frac{\partial y}{\partial x}\right)^2\right]\Delta x$$

$$y(x, t) = Af(kx - \omega t)$$

$$\Delta E = \frac{1}{2}\rho A^2[(\omega f'(kx - \omega t))^2 + s^2(kf'(kx - \omega t))^2]\Delta x = \frac{1}{2}\rho A^2\omega^2[f'(kx - \omega t)]^2 \Delta x$$

The last expression satisfies the necessary conditions.

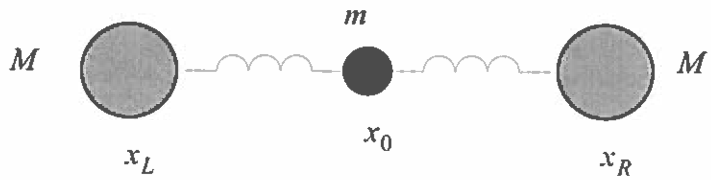
d) The string vibrates in the second normal mode with amplitude A. Write an expression for the total energy E in the string. Make the expression as simple as possible. (You can leave the expression in terms of an integral if you'd like.)

$$y(x, t) = A\cos(\omega_2 t)\sin(k_2 x) = A\cos\left(\frac{2\pi s}{L}t\right)\sin\left(\frac{2\pi}{L}x\right)$$

$$E = \int_0^L dE = \frac{1}{2}\rho A^2\int_0^L\left[\left\{\left(\frac{2\pi s}{L}\right)\sin\left(\frac{2\pi s}{L}t\right)\sin\left(\frac{2\pi}{L}x\right)\right\}^2 + \left\{\left(\frac{2\pi s}{L}\right)\cos\left(\frac{2\pi s}{L}t\right)\cos\left(\frac{2\pi}{L}x\right)\right\}^2\right]dx$$

$$E = \frac{1}{2}\rho A^2\left(\frac{2\pi s}{L}\right)^2\left(\frac{L}{2}\right)\left[\left\{\sin\left(\frac{2\pi s}{L}t\right)\right\}^2 + \left\{\cos\left(\frac{2\pi s}{L}t\right)\right\}^2\right] = \frac{1}{2}\rho\pi^2 A^2 s^2$$

Problem 4 (5+10+15=30 points): A carbon dioxide molecule has the following structure: The bonds



between the atoms are approximated by springs with spring constant k. Assume that the atoms are free to move only in the direction along the long axis connecting the three atoms.

a) How many "normal modes" does this system have? (One of them is rather peculiar.) Explain your answer *briefly.*

There are three masses, so three differential equations, so three normal modes.

b) Define a set of coordinates on the figure above, and write the coupled differential equations that determine the motion of each of the three atoms.

$$M\ddot{x}_L = -k(x_L - x_0)$$
$$m\ddot{x}_0 = -k(x_0 - x_L) - k(x_0 - x_R) = -k(2x_0 - x_L - x_R)$$
$$M\ddot{x}_R = -k(x_R - x_0)$$

Notice that $M\ddot{x}_L + m\ddot{x}_0 + M\ddot{x}_R = 0$.

c) Determine the frequencies of the normal modes, and describe the motions of *at least two* of the normal modes themselves. The determinant of the matrix $\begin{bmatrix} A & C & 0 \\ D & B & D \\ 0 & C & A \end{bmatrix}$ is $A^2B - 2ACD$.

Using $\omega_1^2 = k/M$ and $\omega_2^2 = k/m$ we write

$$\ddot{x}_L = -\omega_1^2 x_L + \omega_1^2 x_0$$
$$\ddot{x}_0 = \omega_2^2 x_L - 2\omega_2^2 x_0 + \omega_2^2 x_R$$
$$\ddot{x}_R = \omega_1^2 x_0 - \omega_1^2 x_R$$

and set the determinant of $\begin{bmatrix} -\omega_1^2 + \omega^2 & \omega_1^2 & 0 \\ \omega_2^2 & -2\omega_2^2 + \omega^2 & \omega_2^2 \\ 0 & \omega_1^2 & -\omega_1^2 + \omega^2 \end{bmatrix}$ equal to zero and solve for ω. This gives

$$(-\omega_1^2 + \omega^2)^2(-2\omega_2^2 + \omega^2) - 2(-\omega_1^2 + \omega^2)\omega_1^2\omega_2^2 = (-\omega_1^2 + \omega^2)\omega^2(\omega^2 - \omega_1^2 - 2\omega_2^2) = 0$$

so the frequencies of the three normal modes are $\omega^2 = \omega_1^2$, 0, and $\omega_1^2 + 2\omega_2^2$. The first is the mode where the outside masses oscillate and the center stays fixed. The second is a simple translation of the center of mass. The third is more complicated.