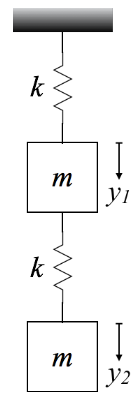
PHYS2063 Wave Physics (Fall 2017)

Midterm Exam #2
Friday 3 Nov 2017

There are **four questions** and you are to work all of them. You are welcome to use your textbook, notes, or any other resources, other than consulting with another human.

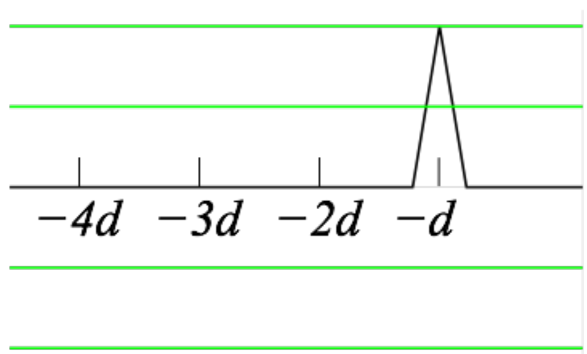
Please start each problem on a new page in your exam booklet.

Good luck!



(1) Two identical masses are connected by identical springs and hang vertically from a fixed point, as shown on the right. The vertical positions of the masses are labeled y_1 and y_2, where $y_1 = 0$ and $y_2 = 0$ are the equilibrium points of the two masses. Write the coupled differential equations of motion, and find the single ("eigen") angular frequencies in terms of $\omega_0 \equiv \sqrt{k/m}$.

(2) At time $t = 0$ a taught string is "plucked" as shown on the right, forming a triangle at $x = -d$. This initial shape is at rest. The string is fixed to $y = 0$ at $x = 0$, but the string has no end to the left. Waves on the string propagate with speed c. Draw the shape of the string for (a) $t = \frac{1}{2}d/c$, (b) $t = d/c$, and (c) $t = 2d/c$. Clearly label the x-position(s) and the size of any shapes you include on your sketch.



(3) A string under tension T and with mass density μ is fixed at $x = 0$ and $x = L$. At time $t = 0$ it lies straight along the x-axis. Also at time $t = 0$ its vertical velocity profile is $v_0(x) = \partial y/\partial t|_{t=0} = V \sin(2\pi x/L)$. Find the shape $y(x, t)$ of the string for all times.

(4) An electric field in some region of space is given by (with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$)

$$\mathbf{E}(x, y, z) = E_0 \left[4\hat{\mathbf{x}} + 2\frac{y}{L}\hat{\mathbf{y}} + 3\left(\frac{z}{L}\right)^2 \hat{\mathbf{z}} \right]$$

Find the total charge, in terms of E_0, L, and ϵ_0, contained in a cube of side length L, with one corner at the origin and lying entirely in the first octant, i.e. $x > 0$, $y > 0$, and $z > 0$.

Solutions

(1) We don't care about gravity since that only shifts the equilibrium position. The equations of motion are $m\ddot{y}_1 = -ky_1 + k(y_2 - y_1) = -2ky_1 + ky_2$ and $m\ddot{y}_2 = -k(y_2 - y_1) = ky_1 - ky_2$. Inserting the ansatz $y_i(t) = C_i \exp(i\omega t)$ and using $\omega_0^2 = k/m$, find $-\omega^2 C_1 = -2\omega_0^2 C_1 + \omega_0^2 C_2$ and $-\omega^2 C_2 = \omega_0^2 C_1 - \omega_0^2 C_2$. Rewrite these as

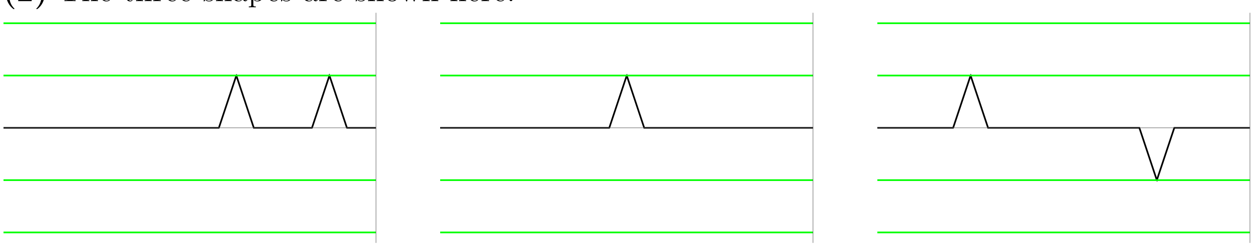
$$\begin{aligned} (\omega^2 - 2\omega_0^2)C_1 + \omega_0^2 C_2 &= 0 \\ \omega_0^2 C_1 + (\omega^2 - \omega_0^2)C_2 &= 0 \end{aligned}$$

we see that we require $(\omega^2 - 2\omega_0^2)(\omega^2 - \omega_0^2) - \omega_0^4 = \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$. Therefore

$$\omega^2 = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2} = \frac{3 \pm \sqrt{5}}{2}\omega_0^2$$

This agrees with the statement of Problem 4.5 (Page 98) in Pain 6e.

(2) The three shapes are shown here:



First, the triangle splits in two, one half moving left and the other moving right. At $t = \frac{1}{2}d/c$ the rightward moving piece is halfway to $x = 0$. At $t = d/c$, the rightward moving pulse collides with its "ghost" to keep the end fixed at $y = 0$. At $t = 2d/c$ the reflection is back at $x = -d$. All the time, the leftward moving pulse just keeps moving to the left.

(3) The general expression for the shape of the string is, with $c \equiv \sqrt{T/\mu}$,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right]$$

Since $y(x,0) = 0$, we must have all $a_n = 0$. Since $\partial y/\partial t|_{t=0} = V\sin(2\pi x/L)$, all $b_n = 0$ except $b_2 = LV/2\pi c$, so

$$y(x,t) = \frac{LV}{2\pi c}\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{2\pi ct}{L}\right)$$

(4) Use Gauss' law with the Divergence Theorem.

$$Q = \epsilon_0 \oint_S \mathbf{E}\cdot\mathbf{dA} = \epsilon_0 \int_C \boldsymbol{\nabla}\cdot\mathbf{E}dV = \epsilon_0 \int_0^L \int_0^L \int_0^L E_0\left[0 + \frac{2}{L} + 6\frac{z}{L^2}\right] dxdydz = 5\epsilon_0 E_0 L^2$$