PHYS2063 Wave Physics (Fall 2017)

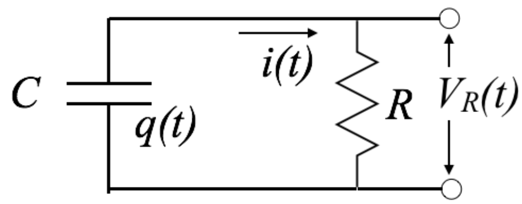
Midterm Exam #1
Friday 29 Sept 2017

There are **four questions** and you are to work all of them. You are welcome to use your textbook, notes, or any other resources, other than consulting with another human.

Please start each problem on a new page in your exam booklet.

Good luck!

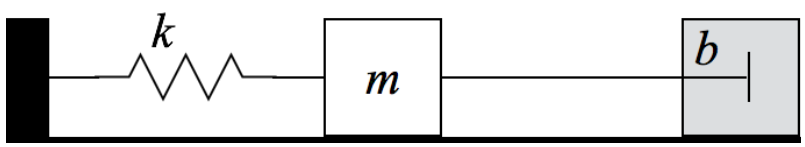
(1) A resistor R and a capacitor C are connected in series:



The charge on the capacitor is $q(t)$ and a current $i(t) = \dot{q}(t)$ flows through the circuit. Write the (first order) differential equation for $q(t)$ and find a solution for it, given $q(0) = q_0$. Use this to find an expression for the voltage $V_R(t)$ across the resistor in terms of R, C, and q_0.

(2) A simple harmonic oscillator is formed from a mass m attached to a spring k on a horizontal, frictionless surface. At time $t = 0$, the mass is located at the equilibrium position, but is moving with a speed v_0. Find the amplitude A of simple harmonic motion in terms of m, k, and v_0.

(3) A mass is attached to a spring and a linear dashpot on a horizontal surface:



For $k = 100$ N/m, $m = 4$ kg, and $b = 32$ N·sec/m, find (a) the (angular) frequency at which the mass oscillates, (b) the period of these oscillations, (c) the time it takes for the energy to decrease by a factor of $1/e$, and (d) the quality factor Q.

(4) In class, we derived the equation of motion for the "forced" oscillator, namely

$$\ddot{x} + 2r\dot{x} + \omega_0^2 x = f_0 \cos \omega t$$

and then found the amplitude $A(\omega)$ for oscillations after the transient motion has died away, in terms of r, ω_0, and f_0. For the weak damping case, that is $r \ll \omega_0$, find the (nonzero to lowest order in r) difference between ω_0 and the frequency ω_M at which the amplitude $A(\omega)$ is a maximum. Be clear whether ω_M is greater or smaller than ω_0.

Solutions

(1) $V_C + V_R = q/C + \dot{q}R = 0$ so $\dot{q} = -q/RC$. The solution is just $q(t) = q_0 \exp(-t/RC)$. Have $V_R(t) = iR = (-q/RC)R = -(q_0/C)\exp(-t/RC)$.

(2) The energy $E = mv_0^2/2 = kA^2/2$ so $A = (m/k)^{1/2}v_0 = v_0/\omega_0$,

(3) Just use the formulas, with $\omega_0^2 = k/m = 25$ and $r = b/2m = 4$.
(a) $\omega_r = (\omega_0^2 - r^2)^{1/2} = (25-16)^{1/2} = 3/\text{sec}$.
(b) $T = 2\pi/\omega_r = 2\pi/3$ sec.
(c) $E = E_0 e^{-2rt}$ so $1/2r = 1/8$ sec.
(d) $Q = \omega_0 m/b = 5 \cdot 4/32 = 5/8$.

(4) We need to set $dA/d\omega = 0$ and solve for ω.

$$\begin{aligned}
\frac{d}{d\omega}\frac{f_0}{\left[(\omega_0^2-\omega^2)^2+4r^2\omega^2\right]^{1/2}} &= -\frac{1}{2}\frac{f_0}{\left[(\omega_0^2-\omega^2)^2+4r^2\omega^2\right]^{3/2}}\left[2(\omega_0^2-\omega^2)(-2\omega)+8r^2\omega\right] = 0 \\
\omega^2 - \omega_0^2 + 2r^2 &= 0 \\
\omega = (\omega_0^2 - 2r^2)^{1/2} &= \omega_0\left(1-\frac{2r^2}{\omega_0^2}\right)^{1/2} \approx \omega_0\left(1-\frac{r^2}{\omega_0^2}\right) \\
\omega - \omega_0 &= -\frac{r^2}{\omega_0}
\end{aligned}$$

That is, the amplitude is a maximum at a frequency just below ω_0. This makes sense because the second term in the denominator of $A(\omega)$ decreases monotonically as you pass through the resonance. I also checked this calculation with MATHEMATICA.